Solution to the exercise problems in Chapter 7

Problem 7.7

$Y(t) = X(t) - X(t-d); \;\; R_X(\tau)$

Find:

a) $R_{X,Y}(\tau); \;\; S_{X,Y}(f)$

b) $R_Y(\tau); \;\; S_Y(f)$

$h(t) = \delta(t) - \delta(t-d)$ - the impulse response

$H(f) = 1 - e^{-j2\pi fd}$

$|H(f)|^2 = (1 - e^{-j2\pi fd})(1 - e^{j2\pi fd}) = 2(1 - \cos 2\pi fd)$

a) $$S_{X,Y}(f) = H(f)^* S_X(f) = S_X(f) - \underbrace{e^{j2\pi fd} S_X(f)}_{\mathcal{F}\{R_X(\cdot+d)\}}$$

$$R_{X,Y}(\tau) = R_X(\tau) - R_X(\tau + d)$$

b) $$S_Y(f) = |H(f)|^2 S_X(f) = \underbrace{2(1 - \cos 2\pi fd)}_{\mathcal{F}\{2\delta(\cdot)-\delta(\cdot+d)-\delta(\cdot-d)\}} \cdot \underbrace{S_X(f)}_{\mathcal{F}\{R_X(\cdot)\}}$$

$$\left[2\cos 2\pi fd = e^{j2\pi fd} + e^{-j2\pi fd} = \mathcal{F}\{\delta(\cdot\, + d) + \delta(\cdot\, - d)\}(f)\right]$$

$$R_Y = R_X * \left(2\delta(\cdot) - \delta(\cdot + d) + \delta(\cdot - d)\right)$$
$$R_Y(\tau) = 2R_X(\tau) - R_X(\tau + d) - R_X(\tau - d)$$

Problem 7.9

$R_X(k) = 4(1/2)^{|k|} + 16(1/4)^{|k|}$

Find $S_X(f)$

- Consider first for $|\alpha| < 1$

$$\sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{-j2\pi fk} = \underbrace{\sum_{-\infty}^{0} \alpha^{-k} e^{-j2\pi fk}}_{-k=k_1 : \sum_0^\infty \alpha^{k_1} e^{j2\pi f k_1}} + \sum_{0}^{\infty} \alpha^{k} e^{-j2\pi fk} - 1$$

$$= \frac{1}{1-\alpha e^{j2\pi f}} + \frac{1}{1-\alpha e^{-j2\pi f}} - 1 =$$

$$= \frac{2 - \alpha(e^{-j2\pi f} + e^{j2\pi f})}{1 + \alpha^2 - \alpha \underbrace{(e^{-j2\pi f} + e^{j2\pi f})}_{2\cos 2\pi f}} - 1$$

$$= \frac{2 - 2\alpha \cos 2\pi f - 1 - \alpha^2 + 2\alpha \cos 2\pi f}{1 + \alpha^2 - 2\alpha \cos 2\pi f} = \boxed{\frac{1-\alpha^2}{1+\alpha^2 - 2\alpha \cos 2\pi f}}$$

-

$$S_X(f) = 4 \cdot \frac{1-(1/2)^2}{1 + 1/4 - 2 \cdot \frac{1}{2} \cos 2\pi f} + 16 \cdot \frac{1-(1/4)^2}{1 + 1/16 - 2 \cdot \frac{1}{4} \cos 2\pi f}$$

$$= \boxed{\frac{12}{5 - 4\cos 2\pi f} + \frac{240}{17 - 8\cos 2\pi f}}$$

Problem 7.19

$$Y(t) = \frac{1}{T} \int_{t-T}^{t} X(t')dt'$$

$\downarrow$

a short-term average of $X(t)$

Find $S_Y(f)$ in terms of $S_X(f)$

- $h(t) = \frac{1}{T} \int_{t-T}^{t} \delta(t')dt' = \begin{cases} \frac{1}{T}, & \text{if} \quad t - T \le 0 < t \Leftrightarrow 0 < t \le T \\ 0, & \text{otherwise} \end{cases}$

- $$\begin{aligned} H(f) &= \frac{1}{T}\int_0^T e^{-j2\pi ft}dt = \\ &= \frac{1}{T}\cdot\frac{1}{j2\pi f}[1-e^{-j2\pi fT}] = \\ &= \frac{1}{T}\cdot\frac{e^{-j\pi fT}}{j2\pi f}[\underbrace{e^{j\pi fT}-e^{-j\pi fT}}_{j2\sin\,\pi fT}] = \boxed{\frac{\sin\,\pi fT}{\pi fT}e^{-j\pi fT}} \end{aligned}$$

- $\boxed{S_Y(f) = |H(f)|^2 S_X(f) = \dfrac{\sin^2\,\pi fT}{(\pi fT)^2}\,S_X(f)}$

Problem 7.34

$$Y_n = \frac{3}{4}Y_{n-1} - \frac{1}{8}Y_{n-2} + W_n \qquad |W_n \text{ -zero mean white-noise proc.} \qquad (*)$$

$\downarrow$

$AR(2)$

a) **Show** that $h_n = \begin{cases} 2\cdot 2^{-n} - 4^{-n}, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$

b) **Find** $H(f)$; $S_Y(f)$; $R_Y(k)$

---

a) The process Y is obtained by passing the white-noise process W through a linear time-invariant system. According to (*), the unit impulse response function h_n of the system is defined as

$$\mathrm{h}_n = \frac{3}{4}\mathrm{h}_{n-1} - \frac{1}{8}\mathrm{h}_{n-2} + \delta_n, \quad n = 0,\ \pm 1,\ \pm 2, \ldots$$

It is easy to check that the functions h_n in a) and h_n are the same.

Below we use a standard technique for computing the unit impuls response function of a system producing an autoregressive process from the white-noise process.

$Y_n - \frac{3}{4}Y_{n-1} + \frac{1}{8}Y_{n-2} = W_n \qquad (*)$

$P(D) = D^\circ - \frac{3}{4}\,D + \frac{1}{8}\,D^2$

$\downarrow$

the polynomial shift operator corresponding to $(*)$

(∗): $\boxed{P(D)(Y) = W}$

To express Y in terms of W, we need the inverse operator $P(D)^{-1}$.

$P(z) = 1 - \frac{3}{4}\,z + \frac{1}{8}\,z^2$ - the characteristic polynomial of $P(D)$

$P(z) = \frac{1}{8}\,(z-2)(z-4) = (1 - z/2)(1 - z/4)$

$$[P(z)]^{-1} = \frac{1}{(1 - z/2)(1 - z/4)} = \frac{2}{1 - z/2} - \frac{1}{1 - z/4}$$

$$= 2\sum_0^\infty \left(\frac{z}{2}\right)^k - \sum_0^\infty \left(\frac{z}{4}\right)^k = \sum_0^\infty \left[2\left(\frac{1}{2}\right)^k - \left(\frac{1}{4}\right)^k\right] z^k = \sum_0^\infty h_k z^k.$$

The inverse operator $P(D)^{-1}$ is now obtained from the above as

$$\boxed{P(D)^{-1} = \sum h_k D^k}$$

$$\left[\left(P(D)^{-1}(\delta)\right)_n = \sum_{k=0}^\infty h_k \delta_{n-k} = \boxed{h_n}\right]$$

b)

$$H(f) = \sum_0^\infty \left(2\left(\frac{1}{2}\right)^k - \left(\frac{1}{4}\right)^k\right) e^{-j2\pi f k}$$
$$= 2\,\frac{1}{1 - \frac{1}{2}e^{-j2\pi f}} - \frac{1}{1 - \frac{1}{4}e^{-j2\pi f}}$$

$$= \frac{1}{(1 - \frac{1}{2}e^{-j2\pi f})(1 - \frac{1}{4}e^{-j2\pi f})}.$$

Then

$$S_Y(f) = |H(f)|^2 \sigma_W^2,$$

where

$$|H(f)|^2 = \frac{1}{1 + \frac{1}{4} - \cos\, 2\pi f} \cdot \frac{1}{1 + \frac{1}{16} - \frac{1}{2}\cos\, 2\pi f}$$

$$= \frac{8}{7}\Big[\frac{2}{1 + \frac{1}{4} - \cos\, 2\pi f} - \frac{1}{1 + \frac{1}{16} - \frac{1}{2}\cos\, 2\pi f}\Big].$$

Recall from Problem 7.9 that for $|\alpha| < 1$

$$\mathcal{F}\{\alpha^{|k|}\}(f) = \frac{1-\alpha^2}{1+\alpha^2 - 2\alpha\cos 2\pi f}.$$

From this and from

$$S_Y(f) = \frac{8}{7}\Big[\frac{2}{1-(\frac{1}{2})^2}\cdot\frac{1-(\frac{1}{2})^2}{1+(\frac{1}{2})^2-\cos 2\pi f} - \frac{1}{1-(\frac{1}{4})^2}\cdot\frac{1-(\frac{1}{4})^2}{1+(\frac{1}{4})^2-\frac{1}{2}\cos 2\pi f}\Big]\sigma_W^2$$

we obtain

$$R_Y(k) = \sigma_W^2\Big[\frac{64}{21}\Big(\frac{1}{2}\Big)^{|k|} - \frac{128}{105}\Big(\frac{1}{4}\Big)^{|k|}\Big].$$

Next we compute $R_Y(k)$ by help of standard technique. For convenience, denote $R(k) = R_Y(k)$.

- $Y_n = \frac{3}{4}Y_{n-1} - \frac{1}{8}Y_{n-2} + W_n$

Multiply both sides by Y_{n-k} to get

$$Y_{n-k}Y_n = \tfrac{3}{4}Y_{n-k}Y_{n-1} - \tfrac{1}{8}Y_{n-k}Y_{n-2} + Y_{n-k}W_n$$

and take the expectation of both sides.

$$R(k) = \frac{3}{4}R(k-1) - \frac{1}{8}R(k-2) + E[Y_{n-k}W_n]$$

$\underline{k=0}$: $\quad R(0) = \frac{3}{4}R(1) - \frac{1}{8}R(2) + \sigma_W^2$

$\underline{k=1}$: $\quad R(1) = \frac{3}{4}R(0) - \frac{1}{8}R(1) \qquad$ (Y_{n-1} and W_n are uncorrelated)

$\underline{k=2}$: $\quad R(2) = \frac{3}{4}R(1) - \frac{1}{8}R(0)$

$\underline{k>2}$: $\quad \boxed{R(k) = \frac{3}{4}R(k-1) - \frac{1}{8}R(k-2)}$

true for $k \geq 2$

- We first find $R(0)$ and $R(1)$ from the first two equations, substituting there

$R(2)$ by the right hand side of the equality with $k = 2$ above.

$$\left| \begin{array}{l} R(0) = \frac{3}{4}R(1) - \frac{1}{8}\left[\frac{3}{4}R(1) - \frac{1}{8}R(0)\right] + \sigma_W^2 \\ \\ R(1) = \frac{3}{4}R(0) - \frac{1}{8}R(1) \end{array} \right.$$

$$\left| \begin{array}{l} \frac{63}{64}R(0) - \frac{21}{32}R(1) = \sigma_W^2 \qquad r_i = \frac{R(i)}{\sigma_W^2}, \quad i = 0, 1 \\ \\ \frac{3}{4}R(0) - \frac{9}{8}R(1) = 0 \end{array} \right.$$

$$\left| \begin{array}{l} 63r_0 - 42r_1 = 64 \\ 6r_0 - 9r_1 = 0 \end{array} \right.$$

$$r_0 = \frac{\det \begin{bmatrix} 64 & -42 & = \\ 0 & -9 \end{bmatrix}}{\det \begin{bmatrix} 63 & -42 \\ 6 & -9 \end{bmatrix}} = \frac{-64 \cdot 9}{-63 \cdot 9 + 6 \cdot 42} = \frac{-576}{-315} = \boxed{\frac{576}{315}}$$

$$r_1 = \frac{\det \begin{bmatrix} 63 & 64 \\ 6 & 0 \end{bmatrix}}{-315} = \boxed{\frac{384}{315}}$$

$$\boxed{R(0) = \frac{576}{315}\sigma_W^2, \quad R(1) = \frac{384}{315}\sigma_W^2}$$

• Now we compute $R(k), \quad k \geq 2$. As we saw above, for $k \geq 2$ the series $R(k)$ obey the recurrent equations

$$R(k) - \frac{3}{4}R(k-1) - \frac{1}{8}R(k-2) = 0, \quad k \geq 2,$$

called second-order difference equations. The characteristic polynomial of the system is

$$P(\lambda) = \lambda^k - \frac{3}{4}\,\lambda^{k-1} + \frac{1}{8}\,\lambda^{k-2}.$$

We need to find the non-zero rooths of this polynomial, i.e., the non-zero rooths of $\lambda^2 - \frac{3}{4}\lambda + \frac{1}{8}$. Easy to see that these rooths are $\lambda_1 = \frac{1}{2}, \ \ \lambda_2 = \frac{1}{4}$. All possible solutions of the system of difference equations have the form

$$\boxed{R(k) = \alpha\left(\frac{1}{2}\right)^k + \beta\left(\frac{1}{4}\right)^k} \ ,$$

where α and β are some constants. We have already computed $R(0)$ and $R(1)$, thus we must have

$$k = 0: \quad \alpha + \beta = R(0),$$

$$k = 1: \quad \frac{\alpha}{2} + \frac{\beta}{4} = R(1).$$

This gives

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}^{-1} \begin{bmatrix} R(0) \\ R(1) \end{bmatrix}$$

$$= \frac{1}{\det\begin{bmatrix} 1 & 1 \\ 1/2 & 1/4 \end{bmatrix}} \begin{bmatrix} 1/4 & -1 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} \frac{576}{315}\sigma_W^2 \\ \frac{384}{315}\sigma_W^2 \end{bmatrix}$$

$$= \frac{(-4)\sigma_W^2}{315} \begin{bmatrix} 144 - 384 \\ -288 + 384 \end{bmatrix}$$

$$= \frac{-4\sigma_W^2}{315} \begin{bmatrix} -240 \\ +96 \end{bmatrix} = \begin{bmatrix} \frac{64}{21}\sigma_W^2 \\ -\frac{128}{105}\sigma_W^2 \end{bmatrix}$$

and hence

$$\boxed{R(k) = \sigma_W^2\left[\frac{64}{21}\left(\frac{1}{2}\right)^{|k|} - \frac{128}{105}\left(\frac{1}{2}\right)^{|k|}\right]}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

Problem 7.44

- $X_\alpha = Z_\alpha + N_\alpha, \quad N_\alpha$ - white noise, $\sigma_N^2 = 1$
- $Z_\alpha = \frac{1}{2} Z_{\alpha-1} + W_\alpha$

- $R_Z(k) = 4(\frac{1}{2})^{|k|}$, $k = 0, \pm 1, \ldots$

a) **Find** the optimum linear filter for estimating Z_α when $p = 1$.

b) **Find** the mean-square error in a)

---

a) $\underline{p = 1}$. We have to solve the system

- $$\begin{bmatrix} 1 \\ r \end{bmatrix} = \begin{bmatrix} 1+\tau^{-1} & r \\ r & 1+\tau^{-1} \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}$$

  with $r = 1/2, \quad \tau = \sigma_Z^2/\sigma_N^2 = 4.$

- $$\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \frac{1}{(1+\tau^{-1})^2 - r^2} \begin{bmatrix} 1+\tau^{-1} & -r \\ -r & 1+\tau^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ r \end{bmatrix}$$

  $$= (\frac{1}{1+\tau^{-1})^2 - r^2} \begin{bmatrix} 1+\tau^{-1} - r^2 \\ r\tau^{-1} \end{bmatrix}$$

- Put $r = 1/2, \quad \tau^{-1} = 1/4$, to get

$$\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} \frac{16}{21} \\ \\ \frac{2}{21} \end{bmatrix}.$$

b) $E[e_\alpha^2] = \sigma_Z^2 - h_0\sigma_Z^2 - h_1 r \sigma_Z^2 = 4(1 - \frac{16}{21} - \frac{2}{21} \cdot \frac{1}{2})$

$$E[e_\alpha^2] = \boxed{\frac{16}{21}}$$