Statistical inference for CIR process (cf. Exercise 6.17 and Section 12.1) dX (t) = b * (a - X (t)) dt + X (t) ^ (1 / 2) dB (t)

```
In[26]:= << Statistics`DescriptiveStatistics`; << Statistics`ContinuousDistributions`;
```

The transition density function for the CIR process pCIR[a, b, x, y, t] = P {X (t) ∈ dy | X (0) = x} dy can be found at various places in the literature :

```
In[29]:= Clear[c1]; c1[b_, t_] := 2 * b / (1 - Exp[-b * t]); Clear[u1];
         u1[b_, x_, t_] := c1[b, t] * x * Exp[-b * t]; Clear[v1];
         v1[b_, y_, t_] := c1[b, t] * y; Clear[pCIR]; pCIR[a_, b_, x_, y_, t_] :=
          c1[b, t] * Exp[-u1[b, x, t] - v1[b, y, t]] * (v1[b, y, t] / u1[b, x, t]) ^ (a * b - 1 / 2) *
           BesselI[2 * a * b - 1, 2 * Sqrt[v1[b, y, t] * u1[b, x, t]]];
```

The stationary distribution of the CIR process is given by (6.64) :

```
In[50]:= Exp[Integrate[2 * b * (a - y) / y, {y, 1, x}, Assumptions → x > 1]] / x
```

$$\text{Out[50]}= \frac{e^{2\,(b-b\,x+a\,b\,\text{Log}[x])}}{x}$$

```
In[51]:= Integrate[%, {x, 0, Infinity}, Assumptions → b > 0 && a > 0]
```

$$\text{Out[51]}= 4^{-a\,b}\, b^{-2\,a\,b}\, e^{2\,b}\, \text{Gamma}[2\,a\,b]$$

```
In[52]:= Clear[PDFStat]; PDFStat[a_, b_, x_] := e^(2 (b-b x+a b Log[x])) / x / (4^(-a b) b^(-2 a b) e^(2 b) Gamma[2 a b]);
```

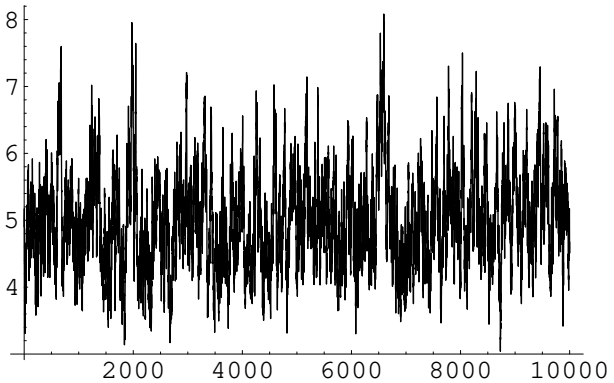
Check that PDFStat satisfies the Forward Equation (6.63) (D_x) ^ 2 (x * p) / 2 + D_x (b * (a - x) * p) = 0 :

```
In[54]:= Simplify[D[D[x * PDFStat[a, b, x], x], x] / 2 - D[b * (a - x) * PDFStat[a, b, x], x]]
```

$$\text{Out[54]}= 0$$

Simulation of an OU process is straightforward by the Euler method :

```
In[65]:= a0 = 5; b0 = 5; StepsPerTimeUnit = 100; T = 100; CIRData =
          {x /. FindRoot[PDFStat[a0, b0, x] == Random[UniformDistribution[0, 1]], {x, 1}]};
         For[i = 2, i ≤ StepsPerTimeUnit * T, i++, AppendTo[CIRData, CIRData[[i - 1]] +
            b0 * (a0 - CIRData[[i - 1]]) / StepsPerTimeUnit + Sqrt[CIRData[[i - 1]]] *
             Random[NormalDistribution[0, 1 / Sqrt[StepsPerTimeUnit]]]]];

In[66]:= ListPlot[CIRData, PlotJoined → True];
```

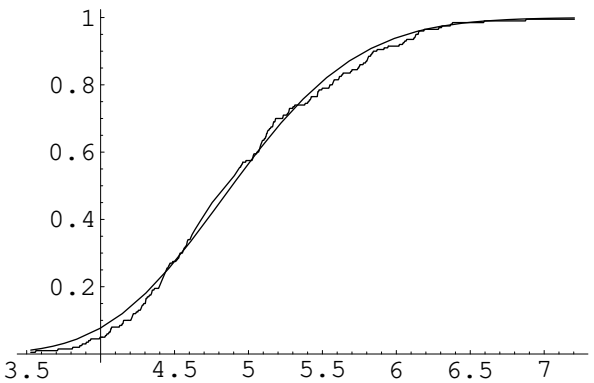


Method 1 : Estimation of mu by Likelihood Ratio according to Equation (10.42)

```
In[84]:= {a1 , b1} =
          {a, b} /. Solve[ {b * Sum[CIRData[[i - 1]] ^ (-1) * (CIRData[[i]] - CIRData[[i - 1]]),
                  {i, 2, StepsPerTimeUnit * T}] - b^2 * a * Sum[CIRData[[i]] ^ (-1) /
                   StepsPerTimeUnit, {i, 1, StepsPerTimeUnit * T}] + b^2 * T == 0,
              a * Sum[CIRData[[i - 1]] ^ (-1) * (CIRData[[i]] - CIRData[[i - 1]]),
                  {i, 2, StepsPerTimeUnit * T}] -
                Sum[CIRData[[i]] - CIRData[[i - 1]], {i, 2, StepsPerTimeUnit * T}] - b *
                 a^2 * Sum[CIRData[[i]] ^ (-1) / StepsPerTimeUnit, {i, 1, StepsPerTimeUnit * T}] +
                2 * b * a * T - b * Sum[CIRData[[i]] / StepsPerTimeUnit,
                  {i, 1, StepsPerTimeUnit * T}] == 0}, {a, b}][[2]]

Out[84]= {4.99511, 5.12804}
```

Method 2 : Estimation of mu by minimizing Kolmogorov - Smirnov Distance between empirical CDF and stationary CDF

```
In[75]:= Clear[FEmp]; FEmp[data_, x_] :=
          {lt = Length[data], Sum[If[data[[i]] ≤ x, 1, 0], {i, 1, lt}] / lt}[[2]];

In[71]:= Simplify[Integrate[PDFStat[a, b, y], {y, 0, x}], Assumptions → a > 0 && b > 0 && x > 0]

Out[71]= 1 - Gamma[2 a b, 2 b x] / Gamma[2 a b]

In[73]:= Clear[CDFStat]; CDFStat[a_, b_, x_] := 1 - Gamma[2 a b, 2 b x] / Gamma[2 a b];

In[76]:= SampleDistance = 50; NData = StepsPerTimeUnit * T / SampleDistance;
         NewCIRData = Table[CIRData[[i * SampleDistance]], {i, 1, NData}];
         SortNewCIRData = Sort[NewCIRData]; {a2, b2} = {a, b} /.
           NMinimize[{Max[Table[Max[{Abs[CDFStat[a, b, SortNewCIRData[[i]]] - (i - 1) / NData],
                  Abs[CDFStat[a, b, SortNewCIRData[[i]]] - i / NData]}],
                {i, 1, NData}]], a > 0, b > 0}, {a, b}][[2]]

Out[76]= {4.91888, 5.49486}

In[77]:= Show[{Plot[FEmp[NewCIRData, x], {x, SortNewCIRData[[1]], SortNewCIRData[[NData]]}],
            Plot[CDFStat[a2, b2, x], {x, SortNewCIRData[[1]], SortNewCIRData[[NData]]}]}];
```



Method 3 : Estimation of mu by Maximum Likelihood method based on pCIR[a, b, x, y, t]

```
In[68]:= {a3, b3} = {a, b} /.
           NMaximize[{Sum[Log[pCIR[a, b, NewCIRData[[i - 1]], NewCIRData[[i]], T / NData]],
               {i, 2, NData}], a > 0, b > 0}, {a, b}][[2]]

Out[68]= {4.94946, 5.58486}
```