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Suggested solution for trialexam in
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1. The three observations 2,3,4 have the same likelihood as one normally distributed observation with value 3, and variance $0.5^2/3 = 1/12$. Thus we have a prior distribution with expectation 4 and precision $1/1^2 = 1$, and a likelihood with observed value 3 and precision 12. The posterior becomes a normal distribution with expectation

$$\frac{4 \cdot 1 + 3 \cdot 12}{1 + 12} = \frac{40}{13} = 3.077$$

   and standard deviation

$$\sqrt{\frac{1}{1+12}} = \sqrt{\frac{1}{13}} = 0.277$$

2. If we are studying the probability distribution for θ, if we know that its density $\pi(\theta)$ is proportional to the function $g(\theta)$, and if $g(\theta)$ is approximated by the density $p(\theta)$ from which we know how to simulate, then importance sampling can be used to find an approximation for the expected value of some function h of the parameter θ. This is done by simulating $\theta_1, \theta_2, \dots, \theta_n$ from the density p, computing the function $h(\theta_i)$ of each of these, and taking the weighted average of these values, with weights $g(\theta_i)/p(\theta_i)$. In contrast, sampling importance resampling is used to find an approximate sample from the distribution π. The θ's and weights are computed as above, but now the weights are used to find probabilities used for a resampling from the θ's, in order to obtain an approximate sample.

3. We get

$$\begin{aligned}
\pi(\lambda \mid \theta) &= 0.8\frac{1}{4}I(|\lambda_1 - \theta_1| \leq 1)I(|\lambda_2 - \theta_2| \leq 1) \\
&\quad +0.2\frac{1}{|2\pi\Sigma|^{1/2}}\exp\left(-\frac{1}{2}\lambda^t\Sigma^{-1}\lambda\right) \\
&= 0.2I(|\lambda_1 - \theta_1| \leq 1)I(|\lambda_2 - \theta_2| \leq 1) \\
&\quad +0.2\frac{1}{2\pi 10}\exp(-\frac{1}{20}(\lambda_1^2 + \lambda_2^2))
\end{aligned}$$

   so for the acceptance probability, we get

$$\begin{aligned}
r &= \min\left(1, \frac{g(\lambda)/\pi(\lambda \mid \theta)}{g(\theta)/\pi(\theta \mid \lambda)}\right) \\
&= \min\left(1, \frac{g(\lambda)\left(20\pi I(|\lambda_1 - \theta_1| \leq 1)I(|\lambda_2 - \theta_2| \leq 1) + \exp(-\frac{1}{20}(\theta_1^2 + \theta_2^2))\right)}{g(\theta)\left(20\pi I(|\lambda_1 - \theta_1| \leq 1)I(|\lambda_2 - \theta_2| \leq 1) + \exp(-\frac{1}{20}(\lambda_1^2 + \lambda_2^2))\right)}\right)
\end{aligned}$$

4. We use the notation in the lecture notes. With the Sidák adjustment, $f_i(T(\theta))$ depends only on $T_i(\theta)$, so if all the test statistics are independent, then so are $f_i(T(\theta))$ for all i, and thus so are all

$$v_i = H_i(\theta)(1 - f_i(T(\theta)))$$

for all i. Also, $\mathrm{E}(f_i(T(\theta))) = \Pr(1 - (1 - T_i(\theta))^N > \alpha) = (1-\alpha)^{1/N}$. So we get

$$\begin{aligned}
FWER &= \Pr(V > 0) \\
&= 1 - \mathrm{E}(I(V = 0)) \\
&= 1 - \mathrm{E}(\prod_{i=1}^{N}(1 - v_i)) \\
&= 1 - \prod_{i=1}^{N}(1 - \mathrm{E}(v_i)) \\
&= 1 - \prod_{i=1}^{N}(1 - H_i(\theta)(1 - (1-\alpha)^{1/N})) \\
&\leq 1 - \prod_{i=1}^{N}(1 - (1 - (1-\alpha)^{1/N})) \\
&= \alpha
\end{aligned}$$

5. (a) For the prior, we have $\pi(p) = 1$, so that

$$\pi(p \mid y) \propto \pi(y \mid p)\pi(p) \propto p_{11}^{y_{11}} p_{12}^{y_{12}} p_{13}^{y_{13}} p_{21}^{y_{21}} p_{22}^{y_{22}} p_{23}^{y_{23}},$$

and the posterior is thus the distribution

$$\text{Dirichlet}(y_{11}+1, y_{12}+1, y_{13}+1, y_{21}+1, y_{22}+1, y_{23}+1).$$

For the prior predictive distribution, we get

$$\begin{aligned}
\pi(y) &= \frac{\pi(y \mid p)\pi(p)}{\pi(p \mid y)} \\
&= \frac{\text{Multinomial}(y \mid p)}{\text{Dirichlet}(p \mid y_{11}+1, y_{12}+1, \ldots, y_{23}+1)} \\
&= \frac{\frac{\Gamma(y_{..}+1)}{\Gamma(y_{11}+1)\Gamma(y_{12}+1)\ldots\Gamma(y_{23}+1)} p_{11}^{y_{11}} \cdots p_{23}^{y_{23}}}{\frac{\Gamma(y_{..}+6)}{\Gamma(y_{11}+1)\Gamma(y_{12}+1)\ldots\Gamma(y_{23}+1)} p_{11}^{y_{11}} \cdots p_{23}^{y_{23}}} \\
&= \frac{\Gamma(y_{..}+1)}{\Gamma(y_{..}+6)}.
\end{aligned}$$

(b) Again, we have that $\pi(\theta) = 1$, so

$$\begin{aligned}
\pi(\theta \mid y) &\propto \pi(y \mid \theta)\pi(\theta) \\
&\propto (r_1 s_1)^{y_{11}} (r_1 s_2)^{y_{12}} (r_1 s_3)^{y_{13}} (r_2 s_1)^{y_{21}} (r_2 s_2)^{y_{22}} (r_2 s_3)^{y_{23}} \\
&\propto r_1^{y_{1\cdot}} r_2^{y_{2\cdot}} s_1^{y_{\cdot 1}} s_2^{y_{\cdot 2}} s_3^{y_{\cdot 3}},
\end{aligned}$$

so that the posterior is the product of the Dirichlet$(y_{1\cdot}+1, y_{2\cdot}+1)$ distribution and the Dirichlet$(y_{\cdot 1}+1, y_{\cdot 2}+1, y_{\cdot 3}+1)$ distribution.

For the prior predictive distribution, we get

$$\begin{aligned}
\pi(y) &= \frac{\pi(y \mid \theta)\pi(\theta)}{\pi(\theta \mid y)} \\
&= \frac{\text{Multinomial}(y \mid \theta)}{\text{Dirichlet}(\theta \mid y_{1\cdot}+1, y_{2\cdot}+1)\,\text{Dirichlet}(\theta \mid y_{\cdot 1}+1, y_{\cdot 2}+1, y_{\cdot 3}+1)} \\
&= \frac{\frac{\Gamma(y_{\cdot\cdot}+1)}{\Gamma(y_{11}+1)\Gamma(y_{12}+1)\ldots\Gamma(y_{23}+1)} r_1^{y_{1\cdot}} r_2^{y_{2\cdot}} s_1^{y_{\cdot 1}} s_2^{y_{\cdot 2}} s_3^{y_{\cdot 3}}}{\frac{\Gamma(y_{\cdot\cdot}+2)}{\Gamma(y_{1\cdot}+1)\Gamma(y_{2\cdot}+1)} r_1^{y_{1\cdot}} r_2^{y_{2\cdot}} \frac{\Gamma(y_{\cdot\cdot}+3)}{\Gamma(y_{\cdot 1}+1)\Gamma(y_{\cdot 2}+1)\Gamma(y_{\cdot 3}+1)} s_1^{y_{\cdot 1}} s_2^{y_{\cdot 2}} s_3^{y_{\cdot 3}}} \\
&= \frac{\Gamma(y_{\cdot\cdot}+1)\Gamma(y_{1\cdot}+1)\Gamma(y_{2\cdot}+1)\Gamma(y_{\cdot 1}+1)\Gamma(y_{\cdot 2}+1)\Gamma(y_{\cdot 3}+1)}{\Gamma(y_{\cdot\cdot}+2)\Gamma(y_{\cdot\cdot}+3)\Gamma(y_{11}+1)\Gamma(y_{12}+1)\ldots\Gamma(y_{23}+1)}.
\end{aligned}$$

(c) For the Bayes Factor, we get

$$\begin{aligned}
\text{BF} &= \frac{\pi(y \mid M_D)}{\pi(y \mid M_I)} \\
&= \frac{\Gamma(y_{\cdot\cdot}+2)\Gamma(y_{\cdot\cdot}+3)\Gamma(y_{11}+1)\Gamma(y_{12}+1)\ldots\Gamma(y_{23}+1)}{\Gamma(y_{\cdot\cdot}+6)\Gamma(y_{1\cdot}+1)\Gamma(y_{2\cdot}+1)\Gamma(y_{\cdot 1}+1)\Gamma(y_{\cdot 2}+1)\Gamma(y_{\cdot 3}+1)}
\end{aligned}$$

(d) For the given data, we get

$$\begin{aligned}
\text{BF} &= \frac{\Gamma(6)\Gamma(7)\Gamma(1)\Gamma(2)\Gamma(1)\Gamma(2)\Gamma(1)\Gamma(3)}{\Gamma(10)\Gamma(2)\Gamma(4)\Gamma(2)\Gamma(2)\Gamma(3)} \\
&= \frac{4 \cdot 5}{7 \cdot 8 \cdot 9} = \frac{5}{136} = \frac{1}{27.2}
\end{aligned}$$