Quadratic variation of Brownian motion

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1 Quadratic variation of Brownian motion

Consider the definition of quadratic variation of one-dimensional Brownian motion over the interval [0,t]:

$$[B, B]_t = \lim_{n\to\infty} \sum_{k=1}^{n} |B(t_k^n) - B(t_{k-1}^n)|^2.$$

Apparently, the interesting quantities are the squared increments $|B(t_k^n) - B(t_{k-1}^n)|^2$ of Brownian motion. What can we say about them? Well, we know a bit about the increments of Brownian motion, since the definition of Brownian motion concerns properties of its increments. For example, we know that they are normally distributed with mean zero and variance $t_k^n - t_{k-1}^n$, i.e.,

$$\mathbb{E}\{|B(t_k^n) - B(t_{k-1}^n)|^2\} = t_k^n - t_{k-1}^n.$$

Thus we expect the quadratic variation $[B, B]_t$ to be something like

$$[B, B]_t \approx \lim_{n\to\infty} \sum_{k=1}^{n} (t_k^n - t_{k-1}^n) = t. \qquad (1)$$

The fact that we actually have equality in (1) is one of the most fundamental reasults in stochastic calculus with respect to Brownian motion.

Theorem 1. *Let $\{[B, B]_t\}_{t\geqslant 0}$ denote the quadratic variation process of one-dimensional Brownian motion, B. Then*

$$\mathbb{P}\{\forall\, t \in [0, \infty),\, [B, B]_t = t\} = 1.$$

The proof which is presented below is very detailed so as to allow the reader to understand every step of the way. The stochastic calculus presented in this summer course can be said to rest on three pillars:

1. The quadratic variation of Brownian motion;
2. The construction of the stochastic integral;
3. The Itô Formula.

All three of these will receive a thorough treatment during the summer. We now present the first of these treatments: The proof of the quadratic variation.

Proof. To begin, we will show that

$$\text{for every } t \in [0,\infty), \mathbb{P}\{[B,B]_t = t\} = 1.$$

For an arbitrary $t \in [0,\infty)$, consider a sequence $\pi(t) = \{\pi_n(t)\}_{n=1}^{\infty}$ of partitions of the interval $[0,t]$,

$$\pi_n(t) : 0 = t_0^n < t_1^n < \cdots < t_n^n = t,$$

such that $\max_{1\leqslant k\leqslant n}(t_k^n - t_{k-1}^n) \to 0,$ as $n \to \infty$. With each partition, $\pi_n(t)$, we associate a random variable, S_n, defined by

$$S_n = \sum_{k=1}^{n} |B(t_k^n) - B(t_{k-1}^n)|^2.$$

These random variables are such that $\lim_{n\to\infty} S_n = [B,B]_t$. We want to show that $[B,B]_t = t$, which obviously amounts to showing that $t = \lim_{n\to\infty} S_n$.

Thus it becomes imperative to investigate the events

$$\{\omega \in \Omega : |S_n(\omega) - t| \geqslant \varepsilon\},$$

for any $\varepsilon > 0$. In order for S_n to converge to t, these events need to be associated with low probabilities, i.e. $\mathbb{P}\{|S_n(\omega) - t| \geqslant \varepsilon\} \approx 0$.

By the *Chebychev inequality,* [1] these probabilities can be estimated by

$$\mathbb{P}\{|S_n - t| \geqslant \varepsilon\} \leqslant \frac{1}{\varepsilon^2}\mathbb{E}\{|S_n - t|^2\} = \frac{1}{\varepsilon^2}\mathrm{Var}\{S_n\},$$

since $t = \mathbb{E}\{S_n\}$, as the following calculation shows.

$$\mathbb{E}\{S_n\} = \mathbb{E}\left\{ \sum_{k=1}^{n} |B(t_k^n) - B(t_{k-1}^n)|^2 \right\} = \sum_{k=1}^{n} \mathbb{E}\{|B(t_k^n) - B(t_{k-1}^n)|^2\}$$
$$= \sum_{k=1}^{n} (t_k^n - t_{k-1}^n) = t_n^n - t_0^n = t.$$

As our next order of business we need to investigate the variance $\mathrm{Var}\{S_n\}$. Due to the independent increments of Brownian motion, the terms, $|B(t_k^n) - B(t_{k-1}^n)|^2$, of S_n are independent random variables.

[1] The Chebychev inequality: Let X be any random variable such that $\mathbb{E}\{|X|^2\} < \infty$. Then for any $a \in \mathbb{R}$ and any $\varepsilon > 0$,

$$\mathbb{P}\{|X - a| \geqslant \varepsilon\} \leqslant \frac{1}{\varepsilon^2}\mathbb{E}\{|X - a|^2\}.$$

Proof.

$$\mathbb{E}\{|X - a|^2\} = \mathbb{E}\{|X - a|^2 1_{\{|X-a|\geqslant\varepsilon\}}\} + \mathbb{E}\{|X - a|^2 1_{\{|X-a|<\varepsilon\}}\}$$
$$\geqslant \mathbb{E}\{|X - a|^2 1_{\{|X-a|\geqslant\varepsilon\}}\} \geqslant \varepsilon^2\mathbb{E}\{1_{\{|X-a|\geqslant\varepsilon\}}\} = \varepsilon^2\mathbb{P}\{|X - a| \geqslant \varepsilon\}.$$

□

Hence, [2]

$$\begin{aligned}
&\mathrm{Var}\{S_n\} = \sum_{k=1}^{n} \mathrm{Var}\{|B(t_k^n) - B(t_{k-1}^n)|^2\} \\
&= \sum_{k=1}^{n} \mathbb{E}\{|B(t_k^n) - B(t_{k-1}^n)|^4\} - (\mathbb{E}\{|B(t_k^n) - B(t_{k-1}^n)|^2\})^2 \\
&= \sum_{k=1}^{n} 3(t_k^n - t_{k-1}^n)^2 - (t_k^n - t_{k-1}^n)^2 = 2\sum_{k=1}^{n}(t_k^n - t_{k-1}^n)^2 \leqslant 2t \max_{1\leqslant k\leqslant n}(t_k^n - t_{k-1}^n).
\end{aligned}$$

We have thus obtained the following estimate of the probability $\mathbb{P}\{|S_n - t| \geqslant \varepsilon\}$:

$$\mathbb{P}\{|S_n - t| \geqslant \varepsilon\} \leqslant 2t\varepsilon^{-2} \max_{1\leqslant k\leqslant n}(t_k^n - t_{k-1}^n). \tag{2}$$

Since the partitions $\pi_n(t)$ associated with the random variables S_n are such that $\max_{1\leqslant k\leqslant n}(t_k^n - t_{k-1}^n) \to 0$, as $n \to \infty$, we see that the probabilities

$$\mathbb{P}\{|S_n - t| \geqslant \varepsilon\} \to 0, \text{ as } n \to \infty.$$

This result, however, is not sufficient for our purposes since it does not preclude the existence of sequences $\{S_n\}_{n=1}^{\infty}$ jumping in and out of the interval $(t-\varepsilon, t+\varepsilon)$:

$$|S_n - t| \geqslant \varepsilon, |S_{n+1} - t| \leqslant \varepsilon, |S_{n+2} - t| \geqslant \varepsilon, \ldots;$$

whereas we want the sequences to stay inside the interval $(t - \varepsilon, t + \varepsilon)$ for *all* sufficiently large values of n. We just need to work a little bit harder to obtain our desired result.

For this reason we turn to one of the *Borel-Cantelli lemmata*:
Let $\{A_n\}_{n=1}^{\infty}$ be any sequence of events such that $\sum_{n=1}^{\infty} \mathbb{P}\{A_n\} < \infty$. Then

$$\mathbb{P}\{\textit{All but finitely many of the events } A_n^c \textit{ occur.}\} = 1.$$

If we define the events

$$A_n = \{\omega \in \Omega : |S_n(\omega) - t| \geqslant \varepsilon\},$$

then by the Borel-Cantelli lemma all we need is to have

$$\sum_{n=1}^{\infty} \mathbb{P}\{A_n\} = \sum_{n=1}^{\infty} \mathbb{P}\{|S_n - t| \geqslant \varepsilon\} < \infty.$$

But we already have at our disposal an estimate (2) of $\mathbb{P}\{|S_n - t| \geqslant \varepsilon\}$ that might just be sufficient for the sum to be finite:

$$\sum_{n=1}^{\infty} \mathbb{P}\{A_n\} \leqslant 2t\varepsilon^{-2} \sum_{n=1}^{\infty} \max_{1\leqslant k\leqslant n}(t_k^n - t_{k-1}^n).$$

[2] We use the fact that if X is a normally distributed random variable with zero mean and variance $t - s$, then $\mathbb{E}\{X^4\} = 3(t-s)^2$ and $\mathbb{E}\{X^2\} = t - s$.

If we choose our sequence of partitions, $\{\pi_n(t)\}_{n=1}^{\infty}$, in such a way so as to make the sum $\sum_{n=1}^{\infty} \max_{1\leqslant k\leqslant n}(t_k^n - t_{k-1}^n)$ finite, then we are done. [3] Thus we have obtained that for every $\varepsilon > 0$,

$$\mathbb{P}\{\text{For all but finitely many } n, \text{ the events } \{|S_n - t| < \varepsilon\} \text{ occur.}\} = 1.$$

This is *still* not the result that we want, since we want to show that

$$\mathbb{P}\{\text{For every } \varepsilon > 0 \text{ and for all but finitely many } n, \{|S_n - t| < \varepsilon\}.\} = 1.$$

In order to obtain this, we take a sequence of rational numbers $\varepsilon_m \downarrow 0$ and for every such rational number, ε_m, we define the set

$$B_m \equiv \{\text{For all but finitely many } n, \text{ the events } \{|S_n - t| < \varepsilon_m\} \text{ occur.}\}$$

We know that $\mathbb{P}\{B_m\} = 1$ for every $m \geqslant 1$ and that

$$B_m \downarrow B \equiv \{\text{The sequence } S_n \text{ converges to } t \text{ as } n \to \infty\},$$

because $\varepsilon_m \downarrow 0$. Since $\mathbb{P}\{B_m\} \to \mathbb{P}\{B\}$ if $B_m \downarrow B$, we have obtained the result that $\mathbb{P}\{B\} = 1$, i.e.,

$$\mathbb{P}\{[B,B]_t = t\} = 1.$$

Recall that we considered an arbitrary choice of $t \in [0,\infty)$. Hence, we have shown that

$$\text{For any } t \in [0,\infty), \mathbb{P}\{[B,B]_t = t\} = 1.$$

All that remains for us to complete the proof of the quadratic variation of Brownian motion, is to show that

$$\mathbb{P}\{\text{For every } t \in [0,\infty), [B,B]_t = t\} = 1.$$

Suppose that the opposite is true, i.e.

$$\mathbb{P}\{\text{For every } t \in [0,\infty), [B,B]_t = t\} < 1.$$

Define the event

$$M \equiv \{\omega \in \Omega : \text{ There exists a } t \in [0,\infty) \text{ such that } [B,B]_t(\omega) \neq t.\}.$$

Then $\mathbb{P}\{M\} > 0$, because

$$\mathbb{P}\{M^c\} = \mathbb{P}\{\text{For every } t \in [0,\infty), [B,B]_t = t\} < 1.$$

Choose an $\omega_0 \in M$ and consider what this implies. For this ω_0 we have $[B,B]_t(\omega_0) \neq t$ for some $t \in [0,\infty)$. [4] There exists a sequence $\{t_n\}_{n=1}^{\infty}$ of rational numbers in $[0,\infty)$ such that $t_n \to t$. For each of these rational numbers we know that $[B,B]_{t_n}(\omega_0) = t_n$ and since the quadratic variation process $[B,B]_s$ is continuous as a function of s, [5] we have

$$[B,B]_{t_n}(\omega_0) \to [B,B]_t(\omega_0).$$

[3]An example of such a sequence $\pi(t)$ is where $\max_{1\leqslant k\leqslant n}(t_k^n - t_{k-1}^n) \leqslant t2^{-n}$.

[4]Recall that any real number can be approximated arbitrarily well by a rational number, in the sense that if $x \in \mathbb{R}$ then there exists a sequence of rational numbers $\{x_n\}_{n=1}^{\infty} \in \mathbb{Q}$ such that $x_n \to x$.

[5]Because the Brownian motion, B_s, is continuous then so is the quadratic variation $[B,B]_s$.

But then we have obtained the result that

$$[B, B]_{t_n}(\omega_0) = t_n \to t$$

and

$$[B, B]_{t_n}(\omega_0) \to [B, B]_t(\omega_0) \neq t,$$

which obviously is a contradiction. Thus we are forced to conclude that $\mathbb{P}\{M\} = 0$ i.e.,

$$\mathbb{P}\{\text{For every } t \in [0, \infty), [B, B]_t = t\} = 1.$$

□

A consequence of this theorem is that the paths of Brownian motion are of infinite variation on any interval $[0, t]$. The fact that the integral $\int_0^t B(s, \omega)\, dB(s, \omega)$ does not exists as a Stieltjes integral for every $\omega \in \Omega$, is due to the fact that sample paths of Brownian motion have infinite variation on $[0, t]$; a fact which we have seen stems from our theorem on the quadratic variation of Brownian motion. The desire to build an integration theory for stochastic processes that accommodates for example Brownian motion, has resulted in the subject of *stochastic calculus*.