Stochastic integral process

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1 The Ito integral process

We have often emphasised the fact that the stochastic integral $I(f)$ is a random variable. In this section we shall try to construct a *stochastic process* out of the individual random variables.

For this purpose we pick an $f \in \mathcal{H}^2$ arbitrarily and we let $T > 0$ be a *fixed* point of $[0, \infty)$. Then we know that there exists a unique random variable

$$I(f) = \int_0^T f(\cdot, s)\, dB_s$$

in $L^2(d\mathbb{P})$ which we call a stochastic integral.

If we introduce a collection $\{m_t\}_{t\in[0,T]}$ of *truncation functions* $m_t : \Omega \times [0,T] \to \mathbb{R}$ defined by

$$m_t(\omega, s) = \begin{cases} 1, & s \in [0,t]; \\ 0, & s \notin [0,t], \end{cases}$$

then the stochastic process $m_t f \in \mathcal{H}^2$, as soon as $f \in \mathcal{H}^2$. Thus there exists a unique random variable

$$I(m_t f) = \int_0^T m_t(\cdot, s) f(\cdot, s)\, dB_s \equiv \int_0^t f(\cdot, s)\, dB_s$$

in $L^2(d\mathbb{P})$. Thus, the problem of constructing a stochastic integral *process* seems to be solved by simply defining the process to be the collection $\{I(m_t f)\}_{t\in[0,T]}$ corresponding to the collection of truncation functions $\{m_t\}_{t\in[0,T]}$, right?

WRONG!

Why, what is wrong with this construction?

The answer is that the stochastic integral $I(f)$ is a very fuzzy object due to the problem of uniqueness with $L^2(d\mathbb{P})$-defined objects as we discussed in Note **??**.

The fact that $I(m_t f)$ exists as a unique random variable in $L^2(d\mathbb{P})$ implies that on subsets $A_t \subseteq \Omega$ such that $\mathbb{P}\{A_t\} = 0$, $I(m_t f)$ is not defined in a unique way. Since $\mathbb{P}\{A_t\}$ exists, each set, A_t,is $\mathcal{F}_T$-measurable, i.e., $A_t \in \mathcal{F}_T$. Thus we have an *uncountable* collection $\{A_t\}_{t\in[0,T]}$ of elements in the sigma-algebra $\mathcal{F}_T$. If we consider their union

$$A = \bigcup_{t\in[0,T]} A_t,$$

we cannot tell if $A \in \mathcal{F}_T$ or not. It might actually be possible that $A = \Omega$, which implies that "For every $\omega \in \Omega$, there exist at least one point $t \in [0,T]$ such that $I(m_t f)$ is not a random variable. (A random variable has to be defined in a unique way, and $I(m_t f)$ is not defined in a unique way.)"

Thus it might be possible that there is at least one $t \in [0,T]$ that prevents the collection $\{I(m_t f)\}_{t\in[0,T]}$ from being a stochastic process. An easy way out would of course have been to disregard such points $t \in [0,T]$, but the problem is that we don't know where they are. Neither do we know for a fact if indeed $A = \Omega$. All these uncertainties make us try a different approach in finding The Stochastic Integral Process,

$$\left\{ \int_0^t f(\cdot, s)\, dB_s \right\}_{t\in[0,T]}.$$

The following theorem gets us as close as we need to get in order to have a stochastic integral process which we can use in stochastic calculus. The point is that we do not need a stochastic integral $I(m_t f)(\omega)$ to be defined for *every* $\omega \in \Omega$, only for *sufficiently many* $\omega \in \Omega$, i.e., for $\omega \in M$, where $M \subseteq \Omega$ is a set of probability one, $\mathbb{P}\{M\} = 1$. Here then is the theorem.

Theorem 1 (Stochastic integral process). *For any stochastic process $f \in \mathcal{H}^2$, there exists a continuous martingale $\{X_t\}_{t\in[0,T]}$ with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, such that*

$$\text{for every } t \in [0,T],\ \mathbb{P}\{\, X_t = I(m_t f)\,\} = 1.$$

The filtration, $\{\mathcal{F}_t\}_{t\in[0,T]}$, is the natural filtration generated by the Brownian motion, i.e., $\mathcal{F}_t = \sigma(\{B_s : s \leqslant t\})$.

This is quite remarkable! Not only do we get our stochastic integral *process*, but it is also a continuous martingale!

The key result of this theorem is the *Doob maximal inequality* which we now present.

Theorem 2 (The Doob maximal inequality). *Let $\{M_n\}_{n=1}^{\infty}$ be a positive sub-martingale and $\varepsilon > 0$ any real number. Then*

$$\mathbb{P}\Big\{ \sup_{0\leqslant k \leqslant n} M_k \geqslant \varepsilon \Big\} \leqslant \frac{1}{\varepsilon^2}\mathbb{E}\{M_n^2\}.$$

The Doob maximal inequality is a considerable improvement of the Chebychev inequality, [1] but then again the Chebychev inequality applies to any random variable M_n such that $\mathbb{E}\{|M_n|^2\} < \infty$, whereas the Doob maximal inequality "only" applies to positive sub-martingales.

Definition 1 (Sub martingale). *A positive* sub-martingale *is a pair* $(\{M_n\}_{n=1}^{\infty}, \{\mathcal{F}_n\}_{n=1}^{\infty})$ *where, for every $1 \leqslant m \leqslant n$,*

- $\mathbb{E}\{|M_n|\} < \infty$*;*

[1] Recall the Chebychev inequality: For any random variable M_n such that $\mathbb{E}\{|M_n|^2\} < \infty$ and for any $\varepsilon > 0$

$$\mathbb{P}\{|M_n| \geqslant \varepsilon\} \leqslant \frac{1}{\varepsilon^2}\mathbb{E}\{|M_n|^2\}.$$

- $0 \leqslant M_n \in \mathcal{F}_n$;
- $\mathbb{E}\{M_n|\mathcal{F}_m\} \geqslant M_m$.

Proof of Theorem 1. Let $f \in \mathcal{H}^2$ be an arbitrary stochastic process. Then, by the Approximation theorem of $\mathcal{H}^2$, there exists a sequence of stochastic processes $\{f_n\}_{n=1}^{\infty} \in \mathcal{H}_0^2$ such that

$$||f_n - f||_{L^2(d\mathbb{P}\times dt)} \to 0 \text{ as } n \to \infty.$$

Let $t \in [0,T]$ be an arbitrary real number. Then, for each of the processes $f_n \in \mathcal{H}_0^2$, the processes $m_t f_n \in \mathcal{H}_0^2$. Since the stochastic integral $I(g)$ is defined explicitly for any stochastic process $g \in \mathcal{H}_0^2$, the stochastic integral $I(m_t f_n)$ is defined explicitly.

Let

$$0 = t_0 < t_1 < \cdots < t_n = T$$

be a partition of the interval $[0,T]$, associated with the stochastic process f_n. Our arbitrary number $t \in [0,T]$ has to lie between some $t_k < t \leqslant t_{k+1}$.

$$X_t^n(\omega) \equiv I(m_t f_n)(\omega) = c_k(\omega)\{B(t) - B(t_k)\} + \sum_{i=1}^{k} c_i(\omega)\{B(t_i) - B(t_{i-1})\}.$$

Since Brownian motion is a continuous martingale with respect to its natural filtration, $\{\mathcal{F}_t\}_{t\in[0,T]}$, so is, for every $n \in \mathbb{N}$, $\{X_t^n\}_{t\in[0,T]}$ a continuous martingale with respect to the same filtration, $\{\mathcal{F}_t\}_{t\in[0,T]}$.

Take any two $m, n \in \mathbb{N}$ and define the continuous stochastic process $M = \{M_t\}_{t\in[0,T]}$ by

$$M_t \equiv |X_t^n - X_t^m|.$$

Since the function $\varphi(x) = |x|$ is convex, by the *Jensen inequality* for conditional expectation [2] we have, for any $s \leqslant t$,

$$\varphi\big(\mathbb{E}\{X_t^n - X_t^m|\mathcal{F}_s\}\big) \leqslant \mathbb{E}\{\varphi(X_t^n - X_t^m)|\mathcal{F}_t\},$$

i.e.,

$$|\mathbb{E}\{X_t^n - X_t^m|\mathcal{F}_s\}| \leqslant \mathbb{E}\{M_t|\mathcal{F}_s\}.$$

But $\{X_t^n - X_t^m\}_{t\in[0,T]}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$, so

$$\mathbb{E}\{X_t^n - X_t^m|\mathcal{F}_s\} = X_s^n - X_s^m,$$

implying that

$$M_s = |X_s^n - X_s^m| = |\mathbb{E}\{X_t^n - X_t^m|\mathcal{F}_s\}| \leqslant \mathbb{E}\{M_t|\mathcal{F}_s\},$$

i.e., M_t is a positive sub-martingale.

[2] The Jensen inequality for conditional expectation: *Let $X : \Omega \to \mathbb{R}$ be any random variable and $\mathcal{F}$ any sigma-algebra of subsets of Ω. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function then*

$$\varphi(\mathbb{E}\{X|\mathcal{F}\}) \leqslant \mathbb{E}\{\varphi(X)|\mathcal{F}\}.$$

We may therefore apply the *Doob maximal inequality* to the process $\{M_t\}_{t\in[0,T]}$ to get

$$\mathbb{P}\left\{\sup_{t\in[0,T]} M_t \geqslant \varepsilon\right\} \leqslant \frac{1}{\varepsilon^2}\mathbb{E}\{M_T^2\},$$

where $\varepsilon > 0$ is an arbitrary real number. Now,

$$\begin{aligned}\mathbb{E}\{M_T^2\} &= \mathbb{E}\{|X_T^n - X_T^m|^2\} = \mathbb{E}\{|I(m_T f_n) - I(m_T f_m)|^2\} = \mathbb{E}\{|I(f_n) - I(f_m)|^2\}\\ &= ||I(f_n - f_m)||_{L^2(d\mathbb{P})} = ||f_n - f_m||_{L^2(d\mathbb{P}\times dt)},\end{aligned}$$

where we have used the Itô isometry for processes $f_n, f_m \in \mathcal{H}_0^2$, linearity of the stochastic integral $I(\cdot)$ and the fact that $m_T g = g$, for *any* stochastic process g.

Recall that the sequence of processes $\{f_n\}_{n=1}^{\infty} \in \mathcal{H}_0^2$ is such that

$$||f_n - f||_{L^2(d\mathbb{P}\times dt)} \to 0 \text{ as } n \to \infty.$$

This implies that

$$||f_n - f_m||_{L^2(d\mathbb{P}\times dt)} \to 0, \text{ as } n, m \to \infty \textit{ independently} \text{ of each other.}$$

Because $n, m \to \infty$ independently of each other, we may fix the value of m and let $n \to \infty$. Then, for every choice of m we get

$$||f_n - f_m||_{L^2(d\mathbb{P}\times dt)} \to 0, \text{ as } n \to \infty.$$

So, if we choose n big enough we can make $||f_n - f_m||_{L^2(d\mathbb{P}\times dt)}$ be as small as we like. Let us say that we want $||f_n - f_m||_{L^2(d\mathbb{P}\times dt)} \leqslant 2^{-3m}$. Then there is an integer, N_m, such that whenever $n \geqslant N_m$ we have $||f_n - f_m||_{L^2(d\mathbb{P}\times dt)} \leqslant 2^{-3m}$. Since this inequality holds for *every* $n \geqslant N_m$, we get

$$\max_{n\geqslant N_m} ||f_n - f_m||_{L^2(d\mathbb{P}\times dt)} \leqslant 2^{-3m}.$$

Every value of m is thus associated with an integer N_m, i.e., we have obtained a sequence $\{N_m\}_{m=1}^{\infty}$. We may choose the values of N_m to be *increasing*, i.e., $N_m < N_{m+1}$ for every $m \in \mathbb{N}$. These considerations allow us to express the result from the Doob maximal inequality as

$$\mathbb{P}\left\{\sup_{t\in[0,T]} |X_t^n - X_t^m| \geqslant \varepsilon\right\} \leqslant \frac{1}{\varepsilon^2}||f_n - f_m||_{L^2(d\mathbb{P}\times dt)}.$$

Since this inequality is valid for *any* $n, m \in \mathbb{N}$ we may choose n and m from the sequence $\{N_k\}_{k=1}^{\infty}$. We choose $m = N_k$ and $n = N_{k+1}$. Thus we get

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,T]} |X_t^{N_{k+1}} - X_t^{N_k}| \geqslant \varepsilon\right\} &\leqslant \frac{1}{\varepsilon^2}||f_{N_k+1} - f_{N_k}||_{L^2(d\mathbb{P}\times dt)}\\ &\leqslant \frac{1}{\varepsilon^2}\max_{n\geqslant N_k} ||f_n - f_{N_k}||_{L^2(d\mathbb{P}\times dt)} \leqslant \frac{2^{-3k}}{\varepsilon^2}.\end{aligned}$$

This inequality holds for *every* $\varepsilon > 0$, we may choose ε so that $\varepsilon^{-2}2^{-3k} = 2^{-k}$, i.e., we choose $\varepsilon = 2^{-k}$. Thus we have the inequality

$$\mathbb{P}\left\{\sup_{t\in[0,T]} |X_t^{N_{k+1}} - X_t^{N_k}| \geqslant 2^{-k}\right\} \leqslant 2^{-k}.$$

Define the events

$$A_k \equiv \{\omega \in \Omega : \sup_{t\in[0,T]} |X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega)| \geqslant 2^{-k}\}.$$

For these events we have the probabilities $\mathbb{P}\{A_k\} \leqslant 2^{-k}$. If we consider the sum $\sum_{k=1}^{\infty} \mathbb{P}\{A_k\}$ we get

$$\sum_{k=1}^{\infty} \mathbb{P}\{A_k\} \leqslant \sum_{k=1}^{\infty} 2^{-k} < \infty.$$

Then we may apply the *Borel-Cantelli lemma* to deduce that the event

$$A \equiv \{\omega \in \Omega : \text{ All but finitely many of the events } A_k^c \text{ occur}\}$$

has probability one. Consequently, for every $\omega \in A$, there exists a finite random integer $C(\omega)$ such that for every $k \geqslant C(\omega)$, the events A_k^c occur, i.e.,

$$\text{for every } k \geqslant C(\omega), \sup_{t\in[0,T]} |X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega)| < 2^{-k}.$$

Recall that for every $n \in \mathbb{N}$, $\{X_t^n\}_{t\in[0,T]}$ is a continuous stochastic process. (It is also a martingale, but we do not need this fact for the moment.) The fact that $\{X_t^n\}_{t\in[0,T]}$ is a continuous process means that for every $\omega \in \Omega$, the map $t \mapsto X_t^n(\omega)$ is a continuous function on the interval $[0,T]$.

The collection, $\mathcal{C}([0,T])$, of *all* continuous functions on the interval $[0,T]$ is a complete normed space, i.e., a *Banach space*, where the norm, $||g||$, is given by

$$||g|| \equiv \sup_{t\in[0,T]} |g(t)|,$$

for any element $g \in \mathcal{C}([0,T])$. We will use the following theorem, valid in any Banach space:

Every absolutely convergent series in a Banach space is convergent.

This implies that if $\{g_n\}_{n=1}^{\infty} \in \mathcal{C}([0,T])$ is a sequence of continuous functions such that

$$\sum_{k=1}^{\infty} ||g_k|| < \infty \quad \text{(Absolutely convergent series)},$$

then $\sum_{k=1}^{\infty} g_k$ is convergent, i.e., it is a continuous function on $[0,T]$.

If we choose any $\omega \in A$, the functions $\{g_k\}_{k=1}^{\infty}$, defined by

$$g_k(t) \equiv X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega),$$

are elements of the Banach space $\mathcal{C}([0,T])$. Then

$$\sum_{k=1}^{m} g_k(t) = \sum_{k=1}^{m} \left\{ X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega) \right\} = X_t^{N_{m+1}}(\omega) - X_t^{N_1}(\omega)$$

and

$$\sum_{k=1}^{\infty} g_k(t) = \lim_{m\to\infty} X_t^{N_{m+1}}(\omega) - X_t^{N_1}(\omega).$$

Now, if $\sum_{k=1}^{\infty} ||g_k|| < \infty$, then $\lim_{m\to\infty} X_t^{N_{m+1}}(\omega) - X_t^{N_1}(\omega)$ is a continuous function on $[0,T]$, i.e., $\lim_{m\to\infty} X_t^{N_{m+1}}(\omega)$ is a continuous function on $[0,T]$. (Recall that we already know that $X_t^{N_1}(\omega)$ is a continuous function.)

All we need to do is to check whether the condition $\sum_{k=1}^{\infty} ||g_k|| < \infty$ is satisfied.

$$\begin{aligned}
\sum_{k=1}^{\infty} ||g_k|| &= \sum_{k=1}^{\infty} \sup_{t\in[0,T]} |X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega)| \\
&= \sum_{k=1}^{C(\omega)-1} \sup_{t\in[0,T]} |X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega)| + \sum_{k=C(\omega)}^{\infty} \sup_{t\in[0,T]} |X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega)| \\
&\leqslant \sum_{k=1}^{C(\omega)-1} \sup_{t\in[0,T]} |X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega)| + \sum_{k=C(\omega)}^{\infty} 2^{-k}.
\end{aligned}$$

Because the functions $t \mapsto X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega)$ are continuous and the interval $[0,T]$ is closed and bounded, the maximum value of $X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega)$ is finite and attained for some $t \in [0,T]$. Thus, each of the terms in the sum $\sum_{k=1}^{C(\omega)-1} \sup_{t\in[0,T]} |X_t^{N_{k+1}}(\omega) - X_t^{N_k}(\omega)|$ are finite and since there is a finite number, $C(\omega)-1$, of terms, the sum is finite. This implies that $\sum_{k=1}^{\infty} ||g_k|| < \infty$ and consequently, for every $\omega \in A$, $\lim_{m\to\infty} X_t^{N_{m+1}}(\omega)$ is a continuous function on $[0,T]$. Denote this function by $X_{(\cdot)}(\omega)$, i.e., the map $t \mapsto X_t(\omega)$ is a continuous function.

Recall that $X_t^n(\omega) \equiv I(m_t f_n)(\omega)$ and define the event

$$A' \equiv \{\omega \in \Omega : \lim_{m\to\infty} I(m_{(\cdot)} f_{N_m})(\omega) \text{ is a continuous function on } [0,T].\}.$$

We have seen that, if $\omega \in A$ then $\omega \in A'$, i.e., $A \subseteq A'$. Since $\mathbb{P}\{A\} = 1$ we get

$$1 = \mathbb{P}\{A\} \leqslant \mathbb{P}\{A'\} \leqslant 1,$$

i.e. $\mathbb{P}\{A'\} = 1$.

For every fixed $t \in [0,T]$, $\{X_t^{N_m}\}_{m=1}^{\infty}$ is a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Brownian motion is defined. Then the object $X_t \equiv \lim_{m\to\infty} X_t^{N_m}$ is also a random variable on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Since this holds for every $t \in [0,T]$ we see that $\{X_t\}_{t\in[0,T]}$ is a collection of random variables on the space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., $\{X_t\}_{t\in[0,T]}$ is a stochastic process.

We shall investigate whether the stochastic process $\{X_t\}_{t\in[0,T]}$ is a martingale with respect to the natural filtration, $\{\mathcal{F}_t\}_{t\in[0,T]}$, of Brownian motion. In order for it to be a martingale is has to satisfy the properties, for every $s \leqslant t \in [0,T]$,

- $\mathbb{E}\{|X_t|\} < \infty$;
- $X_t \in \mathcal{F}_t$;
- $\mathbb{E}\{X_t|\mathcal{F}_s\} = X_s$.

Since for every fixed $t \in [0,T]$, $\{X_t^{N_m}\}_{m=1}^{\infty} \in \mathcal{F}_t$ and $X_t = \lim_{m\to\infty} X_t^{N_m}$, $X_t \in \mathcal{F}_t$.

Next, by the definition of *variance* we have

$$\mathrm{Var}[|X_t|] = \mathbb{E}\{|X_t|^2\} - (\mathbb{E}[|X_t|])^2.$$

Since for every random variable Y, $\mathrm{Var}[Y] \geqslant 0$, we deduce that

$$\mathbb{E}\{|X_t|\} \leqslant (\mathbb{E}\{|X_t|^2\})^{1/2},$$

which shows that if we can establish that X_t is an object in $L^2(d\mathbb{P})$, then $\mathbb{E}\{|X_t|\} < \infty$. We know that for every $\omega \in A'$, $X_t(\omega) = \lim_{m\to\infty} X_t^{N_m}(\omega)$, where each of the random variables $X_t^{N_m}$ are elements of $L^2(d\mathbb{P})$. Since $L^2(d\mathbb{P})$ is complete, the limit $\lim_{m\to\infty} X_t^{N_m}$ is a random variable in $L^2(d\mathbb{P})$. This random variable is defined on the *whole* of Ω, and thus may not coincide with the random variable X_t. But we know that it *does* coincide on the subset A' of Ω and that A' has probability one. Thus we may apply our knowledge on uniqueness in $L^2(d\mathbb{P})$ to realise that, as far as uniqueness in $L^2(d\mathbb{P})$ is concerned, the random variable X_t *is* an element of $L^2(d\mathbb{P})$. Consequently $\mathbb{E}\{|X_t|\} < \infty$.

In order to be able to prove the martingale property of $\{X_t\}_{t\in[0,T]}$, i.e., for any $s \leqslant t$, $\mathbb{E}\{X_t|\mathcal{F}_s\} = X_s$, we will employ the *conditional version of the Dominated Convergence theorem*: *If, for every $n \in \mathbb{N}$, $\forall\, \omega \in \Omega, |Y_n(\omega)| \leqslant V(\omega)$, $\mathbb{E}\{V\} < \infty$ and $\mathbb{P}\{Y_n \to Y\} = 1$, then for any sigma-algebra, $\mathcal{G}$, of subsets of Ω*

$$\mathbb{P}\big\{\mathbb{E}\{Y_n|\mathcal{G}\} \to \mathbb{E}\{Y|\mathcal{G}\}\big\} = 1.$$

For an arbitrary $t \in [0,T]$ we define $Y_n \equiv X_t^{N_n}$ and $Y \equiv X_t$. We know that $\mathbb{P}\{Y_n \to Y\} = 1$. Further, we have the explicit representation of Y_n for every $\omega \in \Omega$, given by

$$Y_n(\omega) = c_k(\omega)\{B_t(\omega) - B_{t_k}(\omega)\} + \sum_{i=1}^{k} c_i(\omega)\{B_{t_i}(\omega) - B_{t_{i-1}}(\omega)\},$$

from which we deduce that

$$|Y_n(\omega)| \leqslant |c_k(\omega)||B_t(\omega) - B_{t_k}(\omega)| + \sum_{i=1}^{k} |c_i(\omega)||B_{t_i}(\omega) - B_{t_{i-1}}(\omega)| \equiv V(\omega).$$

By the *Cauchy-Schwartz inequality* we get

$$\mathbb{E}\{V\}^2 \leqslant \mathbb{E}\{|c_k|^2\}\mathbb{E}\{|B_t - B_{t_k}|^2\} + \sum_{i=1}^{k} \mathbb{E}\{|c_i|^2\}\mathbb{E}\{|B_{t_i} - B_{t_{i-1}}|^2\} < \infty,$$

because $\mathbb{E}\{|c_i|^2\} < \infty$ for every $i \in \mathbb{N}$.

Thus, all the prerequisites are satisfied for us to be able to apply the conditional version of the Dominated Convergence Theorem. We deduce that for every $t \in [0,T]$,

$$\mathbb{P}\big\{\mathbb{E}\{X_t^{N_n}|\mathcal{F}_s\} \to \mathbb{E}\{X_t|\mathcal{F}_s\}\big\} = 1.$$

Let $\omega \in A'$ and $s \leqslant t \in [0,T]$ be arbitrary. Then

$$X_s^{N_n}(\omega) = \mathbb{E}\{X_t^{N_n}|\mathcal{F}_s\}(\omega) \to \mathbb{E}\{X_t|\mathcal{F}_s\}(\omega),$$

whereas, at the same time

$$X_s^{N_n}(\omega) \to X_s(\omega),$$

which by the uniqueness of limits implies that

$$\mathbb{E}\{X_t|\mathcal{F}_s\}(\omega) = X_s(\omega), \text{ for every } \omega \in A'.$$

Since $\mathbb{P}\{A'\} = 1$,

$$\mathbb{P}\{\mathbb{E}\{X_t|\mathcal{F}_s\} = X_s\} = 1,$$

which is all that is required for us to establish that the stochastic process $\{X_t\}_{t\in[0,T]}$ is a continuous martingale.

The final piece of the proof of the theorem demonstrate that for every $t \in [0,T]$ the random variable which we have denote by X_t is equal to the stochastic integral $I(m_t f)$ in the sense of $L^2(d\mathbb{P})$, where $f \in \mathcal{H}^2$ is the process we chose at the beginning of the proof on page 3.

Because the sequence $\{f_n\}_{n=1}^{\infty} \in \mathcal{H}_0^2$ approximates the process f in $L^2(d\mathbb{P} \times dt)$, we know that the sub-sequence $\{f_{N_n}\}_{n=1}^{\infty}$ is such that for every $t \in [0,T]$,

$$||m_t f_{N_n} - m_t f||_{L^2(d\mathbb{P}\times dt)} \to 0, \text{ as } n \to \infty.$$

By employing the Itô isometry for processes in $\mathcal{H}^2$ we find that

$$||I(m_t f_{N_n}) - I(m_t f)||_{L^2(d\mathbb{P})} = ||m_t f_{N_n} - m_t f||_{L^2(d\mathbb{P}\times dt)} \to 0, \text{ as } n \to \infty.$$

Since we have already established the fact that $X_t^{N_n} = I(m_t f_{N_n})$ is such that

$$||X_t^{N_n} - X_t||_{L^2(d\mathbb{P})} \to 0, \text{ as } n \to \infty,$$

by the triangle inequality

$$||I(m_t f) - X_t||_{L^2(d\mathbb{P})} \leqslant ||X_t^{N_n} - X_t||_{L^2(d\mathbb{P})} + ||I(m_t f_{N_n}) - I(m_t f)||_{L^2(d\mathbb{P})} \to 0,$$

which tells us that

$$||I(m_t f) - X_t||_{L^2(d\mathbb{P})} = 0,$$

i.e., $I(m_t f)$ and X_t are equal as elements in the space $L^2(d\mathbb{P})$, and this holds for every $t \in [0,T]$.

Thus

$$\text{for every } t \in [0,T], \mathbb{P}\{I(m_t f) = X_t\} = 1$$

and we know that $\{X_t\}_{t\in[0,T]}$ is a continuous martingale. □

The proof of Theorem 1 has been the longest so far and therefore any reader who has followed the proof to its conclusion is to be commended. We hope that the reader has been able to follow the reasoning in every step of the way, since this was the reason for presenting such a lengthy proof in the first place. Again, we thank the reader for bearing with us this far!

Remark 1. *It is possible to strengthen the conclusion of Theorem 1 to the statement that:* If $f \in \mathcal{H}^2$ then

$$\mathbb{P}\{\text{For every } t \in [0,T], X_t = I(m_t f)\} = 1,$$

by using the same technique as we did to establish that for the quadratic variation of Brownian motion we have

$$\mathbb{P}\{\textit{For every } t \in [0,T], [B,B]_t = t\} = 1.$$

Since the proof of Theorem 1 is long enough, this last part was left out.

Because we now have established that when f is a stochastic process from the space $\mathcal{H}^2$ then the stochastic integral $I(f)$ is a continuous martingale, a whole new world is opened up to us. Indeed we may now develop the theory of stochastic calculus in earnest.

The space $\mathcal{H}^2$ will be our "sandbox" in which we play the game of stochastic calculus. The smaller space $\mathcal{H}_0^2 \subset \mathcal{H}^2$ will be used when we need to have explicit representations of the stochastic integrals. Later on we shall see that, unfortunately, the space $\mathcal{H}^2$ is not large enough if we want to do serious stochastic calculus. For this we shall have to consider a space denoted $L^2_{\text{loc}}(d\mathbb{P})$. The spaces of stochastic calculus are thus

$$\mathcal{H}_0^2 \subset \mathcal{H}^2 \subset L^2_{\text{loc}}(d\mathbb{P}).$$

Let us collect our findings so far. We know the following facts:

- If $T \in [0,\infty)$ is a fixed number and $f \in \mathcal{H}_0^2$ a stochastic process, we have an *explicit construction* of a random variable, $I(f)$, which we call the stochastic integral of f over the interval $[0,T]$ with respect to Brownian motion;
- If $f \in \mathcal{H}_0^2$ the we have the *Itô isometry*: $||I(f)||_{L^2(d\mathbb{P})} = ||f||_{L^2(d\mathbb{P}\times dt)}$;
- Every stochastic process $f \in \mathcal{H}^2$ can be approximated by a sequence of stochastic processes $\{f_n\}_{n=1}^{\infty} \in \mathcal{H}_0^2$ in the sense that $||f_n - f||_{L^2(d\mathbb{P}\times dt)} \to 0$, as $n \to \infty$;
- If $T \in [0,\infty)$ is a fixed number and $f \in \mathcal{H}^2$ a stochastic there exists a unique random variable, $I(f)$, in $L^2(d\mathbb{P})$ called the stochastic integral of f over the interval $[0,T]$ with respect to Brownian motion;
- If $f \in \mathcal{H}^2$ the we have the *Itô isometry*: $||I(f)||_{L^2(d\mathbb{P})} = ||f||_{L^2(d\mathbb{P}\times dt)}$;
- If $f \in \mathcal{H}^2$ is a stochastic process, then there exists a *continuous martingale*, $\{X_t\}_{t\in[0,T]}$ with respect to the natural filtration of Brownian motion, such that $\mathbb{P}\{\text{For every } t \in [0,T],\ X_t = I(m_t f)\} = 1$.

Although we have done considerable work in establishing these results, they are not sufficient for us to be able to do stochastic calculus in the same manner as we do ordinary calculus.

The result which enables us to do ordinary calculus is *The Fundamental Theorem of Calculus*; A result which connects integration and differentiation of a function:

$$F(b) - F(a) = \int_a^b F'(x)\,dx.$$

Without this theorem, any explicit computation of of an integral would have had to be reduced to working from the definition of the integral; Since this is so time consuming, the subject of Ordinary Calculus would probably not have developed at all if it hadn't been for the Fundamental Theorem of Calculus.

The same thing holds for the subject of Stochastic Calculus. The corresponding result to the Fundamental Theorem of Calculus in Stochastic Calculus is *The Itô Formula*. This theorem will be our next goal to establish. But before we do that, let us consider the explicit computation of a stochastic integral process by using the tools we have available at the moment.

(The reason we do this computation is to show just how time consuming it can be if we cannot use "higher properties" of stochastic integrals, like the Itô Formula.)

2 An explicit computation

We shall consider the stochastic integral process $I(m_t f)$ corresponding to the process $f \in \mathcal{H}^2$, defined by

$$\text{for every } s \in [0,T],\ f(\omega, s) = B_s(\omega),$$

i.e., the Brownian motion process.

(The verification that the Brownian motion $B \in \mathcal{H}^2$ is left as an exercise to the reader! We have bigger fish to fry!) Because $B \in \mathcal{H}^2$, we know by the *Approximation Theorem* that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of processes $B_n \in \mathcal{H}_0^2$ such that $||B - B_n||_{L^2(d\mathbb{P}\times dt)} \to 0,$ as $n \to \infty$. These processes are given explicitly as

$$B_n(\omega, t) \equiv \sum_{k=1}^{n} c_k(\omega) 1_{(t_{k-1}, t_k]}(t),$$

where $c_k(\cdot)$ are $\mathcal{F}_{t_{k-1}}$-measurable and $\mathbb{E}\{c_k^2\} < \infty$. Because we are approximating Brownian motion itself and $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the natural filtration of Brownian motion, the requirement that the coefficients $c_k(\cdot)$ be $\mathcal{F}_{t_{k-1}}$-measurable suggest that we should choose

$$c_k(\omega) \equiv B(\omega, t_{k-1}).$$

Then $\mathbb{E}\{c_k^2\} = \mathbb{E}\{B_{t_{k-1}}^2\} = t_{k-1} < \infty$ and the approximations to Brownian motion become

$$B_n(\omega, t) \equiv \sum_{k=1}^{n} B(\omega, t_{k-1}) 1_{(t_{k-1}, t_k]}(t).$$

We know confirm that, indeed, $||B - B_n||_{L^2(d\mathbb{P}\times dt)} \to 0,$ as $n \to \infty$. Before we commence with the calculation, note that since $0 = t_0 < t_1 < \cdots < t_n = T$, and we know that the point t has to lie somewhere in the interval $[0, T]$, there is an interval $(t_{k-1}, t_k]$ which contains the point t. Therefore we have that $1 = \sum_{k=1}^{n} 1_{(t_{k-1}, t_k]}(t)$ and consequently

$$B(\omega, t) = B(\omega, t) \cdot 1 = \sum_{k=1}^{n} B(\omega, t) 1_{(t_{k-1}, t_k]}(t).$$

$$
\begin{aligned}
||B - B_n||_{L^2(d\mathbb{P}\times dt)} &= \int_{\Omega\times[0,T]} |B(\omega,t) - B_n(\omega,t)|^2 d\mathbb{P}(\omega)\times dt \\
&= \int_{\Omega\times[0,T]} \left| \sum_{k=1}^{n} \{B(\omega,t) - B(\omega,t_{k-1})\} 1_{(t_{k-1},t_k]}(t) \right|^2 d\mathbb{P}(\omega)\times dt \\
&= \int_{\Omega\times[0,T]} \sum_{k=1}^{n} |B(\omega,t) - B(\omega,t_{k-1})|^2 1_{(t_{k-1},t_k]}(t)\, d\mathbb{P}(\omega)\times dt \\
&= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}\{|B_t - B_{t_{k-1}}|^2\}\, dt = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t - t_{k-1})\, dt \\
&= \frac{1}{2} \sum_{k=1}^{n} (t_k - t_{k-1})^2.
\end{aligned}
$$

If we choose the *uniform partition* of $[0,T]$, i.e., $t_k \equiv \frac{kT}{n}$, we get

$$
\sum_{k=1}^{n} (t_k - t_{k-1})^2 = \frac{T^2}{n^2} \sum_{k=1}^{n} k^2 < \frac{T^2}{n^2} \sum_{k=1}^{\infty} k^2 = \frac{\pi^2}{6} \frac{T^2}{n^2},
$$

and thus

$$
||B - B_n||_{L^2(d\mathbb{P}\times dt)} < \frac{\pi^2}{6} \frac{T^2}{n^2} \to 0, \text{ as } n \to \infty.
$$

By using the uniform partition of the interval $[0,T]$ we have constructed an approximating sequence in $\mathcal{H}_0^2$ of Brownian motion.

Note 1. *There are many approximation sequences $\{f_n\}_{n=1}^{\infty} \in \mathcal{H}_0^2$ to a process $f \in \mathcal{H}^2$. We have found* one *of these for approximating Brownian motion. That is all we need!*

By using the truncation functions $\{m_t\}_{t\in[0,T]}$ and the fact that $m_t B_n$ is an approximation to $m_t B$, for every $t \in [0,T]$, in the sense that

$$
||m_t B - m_t B_n||_{L^2(d\mathbb{P}\times dt)} \to 0, \text{ as } n \to \infty,
$$

we know that the stochastic integral $I(m_t B)$ is approximated by the sequence $\{I(m_t B_n)\}_{n=1}^{\infty}$ in the sense that

$$
\text{for every } t \in [0,T],\ ||I(m_t B) - I(m_t B_n)||_{L^2(d\mathbb{P})} \to 0, \text{ as } n \to \infty.
$$

Because we have an *explicit* construction of the stochastic integrals $I(m_t B_n)$, we can say a bit more regarding the structure of the stochastic integral $I(m_t B)$.

Because the point t has to lie somewhere in the partition $0 = t_0 < t_1 < \cdots < t_n = T$, there exists an integer $k(t) \in \{0, \ldots, n\}$ such that $t \in [t_{k(t)}, t_{k(t)+1}]$. Then

$$
\begin{aligned}
I(m_t B_n)(\omega) = B(\omega, t_{k(t)})\{B(\omega,t) - B(\omega, t_{k(t)})\} \\
+ \sum_{i=1}^{k(t)} B(\omega, t_{i-1})\{B(\omega, t_i) - B(\omega, t_{i-1})\}.
\end{aligned}
$$

We abbreviate $B(\omega, t_i)$ as B_i and consider the sum $\sum_{i=1}^{k(t)} B_{i-1}\{B_i - B_{i-1}\}$.

By using the elementary fact that for any $x, y \in \mathbb{R}$,

$$2xy = (x+y)^2 - x^2 - y^2,$$

we get

$$2B_{i-1}(B_i - B_{i-1}) = B_i^2 - B_{i-1}^2 - (B_i - B_{i-1})^2,$$

and consequently

$$\begin{aligned}\sum_{i=1}^{k(t)} B_{i-1}(B_i - B_{i-1}) &= \frac{1}{2}\sum_{i=1}^{k(t)}(B_i^2 - B_{i-1}^2) - \frac{1}{2}\sum_{i=1}^{k(t)}(B_i - B_{i-1})^2 \\ &= \frac{B_{k(t)}^2}{2} - \frac{1}{2}\sum_{i=1}^{k(t)}(B_i - B_{i-1})^2\end{aligned}$$

Since we are using a uniform partition of $[0,T]$, we know that $t_{k(t)+1} - t_{k(t)} = \frac{T}{n}$; hence $t_{k(t)} \to t$, as $n \to \infty$. We know also that there exists a set $A \subseteq \Omega$ having probability one, $\mathbb{P}\{A\} = 1$, such that if $\omega \in A$ then every function $t \mapsto B(\omega, t)$ is continuous and the quadratic variation of Brownian motion $t \mapsto [B,B]_t(\omega)$ is such that for every $t \in [0,T]$, $[B,B]_t(\omega) = t$. For every $\omega \in A$, the quadratic variation $[B,B]_t(\omega)$ is defined as the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \{B(\omega, t_i) - B(\omega, t_{i-1})\}^2.$$

These two facts combine to give that for every $\omega \in A$,

$$\sum_{i=1}^{k(t)} B(\omega, t_{i-1})\{B(\omega, t_i) - B(\omega, t_{i-1})\} \to \frac{B(\omega,t)^2}{2} - \frac{t}{2}, \text{ as } n \to \infty.$$

Because Brownian motion has independent increments and $\mathbb{P}\{B(0) = 0\} = 1$, we get

$$\begin{aligned}&\mathbb{E}\Big[\big|\{B(t_{k(t)}) - B(0)\}\{B(t) - B(t_{k(t)})\}\big|^2\Big] = \mathbb{E}\big[|B(t_{k(t)}) - B(0)|^2\big]\mathbb{E}\big[|B(t) - B(t_{k(t)})|^2\big] \\ &= t_{k(t)}(t - t_{k(t)}) \leqslant \frac{tT}{n} \to 0, \text{ as } n \to \infty.\end{aligned}$$

If we gather our achievements so far we see that for all ω in a set of probability one,

$$I(m_t B_n)(\omega) \to \frac{B(\omega,t)^2}{2} - \frac{t}{2}, \text{ as } n \to \infty.$$

Since we also know that on a set of probability one $I(m_t B_n) \to I(m_t B)$, we have, by uniqueness in the space $L^2(d\mathbb{P})$, obtained the result that $I(m_t B) = \frac{1}{2}(B_t^2 - t)$. If we *denote* the stochastic integral $I(m_t B)$ by $\int_0^t B_s \, dB_s$ we have obtained by explicit calculation the representation

$$\int_0^t B_s \, dB_s = \frac{1}{2}(B_t^2 - t). \tag{1}$$

Note 2. *The right-hand side,* $\frac{1}{2}(B_t^2 - t)$*, of equation* (1) *agrees with the left-hand hand side* only *on a set of probability one. This set cannot be specified. All we know is that it exists. Therefore is makes no sense to pick a* certain $\omega \in \Omega$ *and ask whether* $\left(\int_0^t B_s\,dB_s\right)(\omega)$ *equals* $\frac{1}{2}(B_t^2 - t)(\omega)$.

But, *if we consider a* bunch *of* ωs*, then we know that for most of these* ωs *we* do *have equality between the right- and the left-hand sides of equation* (1).

This probably seems confusing, but the apparent contradiction lies at the very heart of probability theory. By its very nature, probability theory does not consider individual points, ω*, in the space* Ω*; it only considers* collections *of points. That is why we needed to introduce the concept of a sigma-algebra of subsets of* Ω*, remember!*

The representation in (1) *is very useful when we want to* simulate *the continuous martingale* $\int_0^t B_s\,dB_s$*, because all we need to do is to simulate Brownian motion* $\{B_t\}_{[0,T]}$ *and compute the right-hand side of* (1) *for each simulation. Then, most of our simulated trajectories of* $\frac{1}{2}(B_t^2 - t)$ *will coincide with the stochastic integral* $\int_0^t B_s\,dB_s$.