Stochastic integral as random variable

Anders Muszta

June 28, 2005

1 Stochastic integral

The purpose of this section is to present a fairly rigorous construction of the Ito integral. The text is written in the hope that the arguments leading up to the stochastic integral will be easy to follow for the participants of the summer course.

We will consider defining the stochastic integral with respect to Brownian motion. This object will be denoted

$$\int_0^T f(\omega, t)\, dB_t,$$

but although the notation implies that the stochastic integral is just an ordinary integral as in ordinary calculus, this is a deception for most stochastic processes $\{f(t,\cdot)\}_{t\in[0,\infty)}$, because in order for the object $\int_0^T f(\omega,t)\, dB_t$ to be a ordinary integral, the process $\{B_t\}_{t\in[0,T]}$ needs to be of finite variation. But as we have seen, $\{B_t\}$ is Brownian motion and Brownian motion has positive quadratic variation, hence it has infinite variation.

Any reasonable theory of integration should have no problem to integrate continuous functions and we shall require no less from the theory of stochastic integration. Thus, we want to show that for any continuous function $g : \mathbb{R} \to \mathbb{R}$ there exists such a thing as a stochastic integral

$$\int_0^T g(B_t)\, dB_t.$$

Before we can reach this goal we have to do a lot of hard work.

Remark 1 (Notation). *Denote by $L^2(d\mathbb{P})$ the collection of random variables $X : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega |X(\omega)|^2\, d\mathbb{P}(\omega) < \infty$$

and by $L^2(d\mathbb{P} \times dt)$ the collection of stochastic processes $g : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\int_{\Omega\times[0,T]} |g(\omega,t)|^2\, (d\mathbb{P} \times\, dt)(\omega,t) < \infty.$$

1.1 Simple stochastic integral

We will first define the stochastic integral for simple processes $\{f(\cdot,t)\}_{t\in[0,T]}$. We will call a process *simple* if it is an element of the space $\mathcal{H}_0^2$ and we will call a process *general* if it is an element of the space $\mathcal{H}^2$. The spaces $\mathcal{H}_0^2$ and $\mathcal{H}^2$ are defined as follows.

Definition 1 (The space of general processes, $\mathcal{H}^2$). *The space $\mathcal{H}^2$ is the collection of stochastic processes $f : \Omega \times [0,T] \to \mathbb{R}$ having the properties:*

- *The map $(\omega,t) \mapsto f(\omega,t)$ is $\mathcal{F}_T \times \mathcal{B}_{[0,T]}$-measurable, where $\mathcal{F}_T = \sigma(\{B_s : s \leqslant T\})$ and $\mathcal{B}_{[0,T]}$ denotes the Borel sigma-algebra of subsets of the interval $[0,T]$;*
- *For every $t \in [0,T]$, the random variable $f(\cdot,t)$ is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t = \sigma(\{B_s : s \leqslant t\})$;*
- *The collection $f = \{f(\cdot,t)\}_{t\in[0,T]}$ of random variables is such that*

$$\int_{\Omega\times[0,T]} |f(\omega,t)|^2 \, (d\mathbb{P} \times \, dt)(\omega,t) < \infty.$$

The third property in this list shows that $\mathcal{H}^2$ is a subset of the space $L^2(d\mathbb{P} \times dt)$.

Our goal in this section will be to construct a stochastic integral for processes in $\mathcal{H}^2$. Before we can accomplish this we will construct a stochastic integral for processes in a subspace of $\mathcal{H}^2$, which we will denote $\mathcal{H}_0^2$. The point is of course that it is easier to do the construction in $\mathcal{H}_0^2$ than it is in $\mathcal{H}^2$.

Definition 2 (The space of simple processes, $\mathcal{H}_0^2$). *Let $\mathcal{H}_0^2$ be the subset of $\mathcal{H}^2$ consisting of stochastic processes $f = \{f(\cdot,t)\}_{t\in[0,T]}$ such that*

- *For every $\omega \in \Omega$ and for every $t \in [0,T]$,*

$$f(\omega,t) = \sum_{k=1}^{n} c_k(\omega) 1_{(t_{k-1},t_k]}(t);$$

- *For every $k \in \{1,\ldots,n\}$, the random variables $c_k(\cdot)$ are $\mathcal{F}_{t_{k-1}}$-measurable;*
- *For every $k \in \{1,\ldots,n\}$, $\mathbb{E}\{c_k^2\} < \infty$.*

As a guide as to how we should define a stochastic integral $\int_0^T f(\omega,t)\,dB_t$ when f is a simple stochastic process, we consider what properties we expect a stochastic integral to possess.

If $f(\omega,t) = 1_{(a,b)}(t)$ for every $\omega \in \Omega$, where $(a,b) \subseteq [0,T]$, i.e., the stochastic process $f = \{f(\cdot,t)\}_{t\in[0,T]}$ is a collection of identical (random) variables $1_{(a,b)}(t)$, then we expect a stochastic integral $\int_0^T f(\omega,t)\,dB_t$ to be

$$\int_0^T 1_{(a,b)}(t)\,dB_t = B_b - B_a.$$

We also expect a stochastic integral to be *linear* in the sense that if f and g are stochastic processes and $a,b \in \mathbb{R}$ are constants, then

$$\int_0^T (af+bg)(\omega,t)\,dB_t = a\int_0^T f(\omega,t)\,dB_t + b\int_0^T g(\omega,t)\,dB_t.$$

Because we *demand* these properties of a stochastic integral and because the way a process $f \in \mathcal{H}_0^2$ is defined, we are *forced* to define a stochastic integral, denoted $I(f)$, for processes f in $\mathcal{H}_0^2$ as

Definition 3 (Stochastic integral for processes in $\mathcal{H}_0^2$). *Let $f \in \mathcal{H}_0^2$ be a simple process. Then the stochastic integral is defined by*

$$I(f)(\omega) \equiv \sum_{k=1}^{n} c_k(\omega)\{B(t_k) - B(t_{k-1})\}.$$

The stochastic integral, $I(f)$, is a *random variable*, since we have assumed the time T to be *fixed*. Thus for every fixed $T \in [0, \infty)$ we get a corresponding random variable $I(f)$. The dependence on T is implicit in the definition of $I(f)$; It is to be found through the partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$:

$$I(f)(\omega) = c_1(\omega)\{B(t_1) - B(0)\} + \cdots + c_n(\omega)\{B(T) - B(t_{n-1})\}.$$

For a random variable $X : \Omega \to \mathbb{R}$, denote by $||X||_{L^2(d\mathbb{P})}$ the expectation

$$||X||_{L^2(d\mathbb{P})} \equiv \int_{\Omega} |X(\omega)|^2 \, d\mathbb{P}(\omega)$$

and for a stochastic process $Y : \Omega \times [0, T] \to \mathbb{R}$ denote by $||Y||_{L^2(d\mathbb{P}\times dt)}$ the integral

$$||Y||_{L^2(d\mathbb{P}\times dt)} \equiv \int_{\Omega\times[0,T]} |X(\omega, t)|^2 \, d\mathbb{P}(\omega) \times dt.$$

The following result is one of two key results in defining a stochastic integral for processes in $\mathcal{H}^2$.

Lemma 1 (The Itô isometry for simple processes). *Let $f \in \mathcal{H}_0^2$ be any simple process. Then*

$$||I(f)||_{L^2(d\mathbb{P})} = ||f||_{L^2(d\mathbb{P}\times dt)}.$$

Proof. Because we are working with simple processes we have explicit constructions of both the stochastic process f and the associated random variable $I(f)$:

$$f(\omega, t) = \sum_{k=1}^{n} c_k(\omega) 1_{(t_{k-1}, t_k]}(t) \text{ and } I(f)(\omega) = \sum_{k=1}^{n} c_k(\omega)\{B(t_k) - B(t_{k-1})\}.$$

This allows us to explicitly compute the norms $||f||_{L^2(d\mathbb{P}\times dt)}$ and $||I(f)||_{L^2(d\mathbb{P})}$. We will proceed to do just this.

We begin by computing the norm of the simple process f, $||f||_{L^2(d\mathbb{P}\times dt)}$. By applying the *Fubini theorem* we may write the integral with respect to the product measure $d\mathbb{P} \times dt$ as a double integral,

$$||f||_{L^2(d\mathbb{P}\times dt)} = \int_{\Omega\times[0,T]} |f(\omega, t)|^2 \, d\mathbb{P}(\omega) \times dt = \int_0^T \left\{ \int_{\Omega} |f(\omega, t)|^2 \, d\mathbb{P}(\omega) \right\} dt.$$

In order to proceed we need to compute the square $|f(\omega, t)|^2$, but this we can do since we have an explicit representation of $f(\omega, t)$;

$$|f(\omega, t)|^2 = \left| \sum_{k=1}^{n} c_k(\omega) 1_{(t_{k-1}, t_k]}(t) \right|^2 = \sum_{k=1}^{n} |c_k(\omega)|^2 1_{(t_{k-1}, t_k]}(t).$$

Thus

$$\begin{aligned} ||f||_{L^2(d\mathbb{P}\times dt)} &= \sum_{k=1}^n \left\{ \int_\Omega |c_k(\omega)|^2 \, d\mathbb{P}(\omega) \right\} \int_0^T 1_{(t_{k-1},t_k]}(t)\, dt \\ &= \sum_{k=1}^n \mathbb{E}\{\, |c_k|^2\}(t_k - t_{k-1}). \end{aligned} \tag{1}$$

We will now show that the norm, $||I(f)||_{L^2(d\mathbb{P})}$, of the random variable $I(f)$ equals $\sum_{k=1}^n \mathbb{E}\{\, |c_k|^2\}(t_k - t_{k-1})$.

$$||I(f)||_{L^2(d\mathbb{P})} = \int_\Omega |I(f)(\omega)|^2 \, d\mathbb{P}(\omega) = \int_\Omega \left| \sum_{k=1}^n c_k(\omega)\{B(t_k) - B(t_{k-1})\} \right|^2 d\mathbb{P}(\omega)$$

$$= \sum_{k=1}^n \int_\Omega |c_k(\omega)|^2 |B(t_k) - B(t_{k-1})|^2 \, d\mathbb{P}(\omega)$$

$$+ 2 \sum_{\substack{i,j=1 \\ i<j}}^n \int_\Omega c_i(\omega) c_j(\omega) \{B(t_i) - B(t_{i-1})\}\{B(t_j) - B(t_{j-1})\}\, d\mathbb{P}(\omega)$$

$$= \sum_{k=1}^n \mathbb{E}\{|c_k|^2 |\Delta_k B|^2\} + 2 \sum_{\substack{i,j=1 \\ i<j}}^n \mathbb{E}\{c_i c_j \Delta_i B \Delta_j B\}$$

where $\Delta_k B \equiv B(t_k) - B(t_{k-1})$. Recall that by definition of a simple stochastic process, $f \in \mathcal{H}_0^2$, the random variable c_k is $\mathcal{F}_{t_{k-1}}$-measurable. Also, $\Delta_k B$ is independent of $\mathcal{F}_{t_{k-1}}$, because $\mathcal{F}_{t_{k-1}} = \sigma(\{B_s : s \leqslant t_{k-1}\})$ contains information about the increments of Brownian motion up to, and including, time t_{k-1} and Brownian motion, B, has independent increments. By using the properties of conditional expectation that

- For any random variable X and for any sigma-algebra $\mathcal{F}$, $\mathbb{E}\{X\} = \mathbb{E}\{\mathbb{E}\{X|\mathcal{F}\}\}$;
- If Y is an $\mathcal{F}$-measurable random variable, then $\mathbb{E}\{XY|\mathcal{F}\} = Y\mathbb{E}\{X|\mathcal{F}\}$,

together with the facts on the random variables c_k and the increments $\Delta_k B$, we obtain

$$\mathbb{E}\{|c_k|^2 |\Delta_k B|^2\} = \mathbb{E}\Big\{ \mathbb{E}\{\, |c_k|^2 |\Delta_k B|^2 \,\big|\, \mathcal{F}_{t_{k-1}} \} \Big\} = \mathbb{E}\Big\{ \, |c_k|^2 \mathbb{E}\{\, |\Delta_k B|^2 \,\big|\, \mathcal{F}_{t_{k-1}} \} \Big\}$$
$$= \mathbb{E}\{\, |c_k|^2 \mathbb{E}\{|\Delta_k B|^2\}\} = \mathbb{E}\{\, |c_k|^2\}\mathbb{E}\{|\Delta_k B|^2\} = \mathbb{E}\{\, |c_k|^2\}(t_k - t_{k-1}).$$

As for the terms $\mathbb{E}\{c_i c_j \Delta_i B \Delta_j B\}$ we note that since $i < j$, $c_i c_j \in \mathcal{F}_{t_{j-1}}$ and $\Delta_i B \in \mathcal{F}_{t_i}$. We know further that $\mathcal{F}_{t_i} \subseteq \mathcal{F}_{t_{j-1}}$ because $\{\mathcal{F}_{t_k}\}_{k=1}^n$ is a filtration and $t_i \leqslant t_{j-1}$. Thus $c_i c_j \Delta_i B \in \mathcal{F}_{t_{j-1}}$ which makes $c_i c_j \Delta_i B$ independent of $\Delta_j B$. Thus

$$\mathbb{E}\{c_i c_j \Delta_i B \Delta_j B\} = \mathbb{E}\{c_i c_j \Delta_i B\}\mathbb{E}\{\Delta_j B\} = 0,$$

because increments, $\Delta_j B$, of Brownian motion has mean zero, i.e., $\mathbb{E}\{\Delta_j B\} = 0$.

We may summarize our computations by stating that

$$||I(f)||_{L^2(d\mathbb{P})} = \sum_{k=1}^n \mathbb{E}\{\, |c_k|^2\}(t_k - t_{k-1}) = ||f||_{L^2(d\mathbb{P}\times dt)},$$

and the proof is complete. □

If we use the notation $I(f) = \int_0^T f(\cdot,t)\,dB_t$, then the Itô isometry can be written

$$\mathbb{E}\Big\{\Big(\int_0^T f(\cdot,t)\,dB_t\Big)^2\Big\} = ||I(f)||_{L^2(d\mathbb{P})} = ||f||_{L^2(d\mathbb{P}\times dt)} = \int_0^T \mathbb{E}\{|f(\cdot,t)|^2\}\,dt.$$

This is the form of the Itô isometry that we will use later on in the course when doing "stochastic calculus".

The form $||I(f)||_{L^2(d\mathbb{P})} = ||f||_{L^2(d\mathbb{P}\times dt)}$ of the Itô isometry is useful when we consider fundamental properties of stochastic integrals such as convergence issues. The most important consequence of the Itô isometry is that if $\{f_n\}_{n=1}^\infty$ is a sequence of stochastic processes in $\mathcal{H}_0^2$ such that $f_n \to f$ in the sense that $||f_n - f||_{L^2(d\mathbb{P}\times dt)} \to 0$ as $n \to \infty$, where $f \in \mathcal{H}_0^2$ is a stochastic process, then the Itô isometry implies

$$||I(f_n) - I(f)||_{L^2(d\mathbb{P})} = ||I(f_n - f)||_{L^2(d\mathbb{P})} = ||f_n - f||_{L^2(d\mathbb{P}\times dt)} \to 0 \text{ as } n \to \infty.$$

Thus the stochastic integral is continuous in the sense that if the stochastic processes f and g are close then the random variables $I(f)$ and $I(g)$ will be close.

In order to extend our construction of the stochastic integral $I(f)$ so that we may consider not only processes $f \in \mathcal{H}_0^2$ but also more general processes $f \in \mathcal{H}^2$, we need a theorem that says that we may approximate any process $f \in \mathcal{H}^2$ by processes in $\mathcal{H}_0^2$. The precise result is this.

Lemma 2 (Approximation of $\mathcal{H}^2$ by $\mathcal{H}_0^2$). *Let $f \in \mathcal{H}^2$ be any stochastic process. Then there exists a sequence of stochastic processes $\{f_n\}_{n=1}^\infty$ in $\mathcal{H}_0^2$ such that*

$$||f_n - f||_{L^2(d\mathbb{P}\times dt)} \to 0, asn \to \infty.$$

Of course, if we would not have had this result then the whole subject of stochastic calculus would not have existed. After all, it is not very interesting to only be able to consider simple processes when one is interested in modelling real-world phenomena by stochastic differential equations, since most signals are continuous by nature and simple processes are piecewise constant. If we *were* to use piecewise constant processes to model real-world signals, then at least we would require an approximation theorem which states that any results we come up with using simple processes will be approximations to some real-world counterparts. Therefore we are very fortunate indeed to have the approximation result of Lemma 2 at our disposal.

1.2 General stochastic integral

With the approximation result of Lemma 2 and the Itô isometry (Lemma 1) we have all the tools needed to construct the stochastic integral $I(f)$ for any stochastic process $f \in \mathcal{H}^2$.

Therefore we pick any stochastic process $f \in \mathcal{H}^2$. Then we use the approximation lemma to get a sequence $\{f_n\}_{n=1}^\infty$ of processes in $\mathcal{H}_0^2$ such that $||f_n - f||_{L^2(d\mathbb{P}\times dt)} \to 0$, as $n \to \infty$. With every simple process $f_n \in \mathcal{H}_0^2$ we may associate the stochastic integral $I(f_n)$, so we have a sequence of random variables $\{I(f_n)\}_{n=1}^\infty$. Because the sequence $\{f_n\}_{n=1}^\infty$ converges to f it is a *Cauchy* *sequence* in $L^2(d\mathbb{P} \times dt)$ in the sense that

$$\text{for any } n, m \geqslant 1, ||f_n - f_m||_{L^2(d\mathbb{P}\times dt)} \to 0, \text{ as } n, m \to \infty.$$

Then we use the Itô isometry to get that the sequence $\{I(f_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence, but in $L^2(d\mathbb{P})$, i.e.,

$$\text{for any } n, m \geqslant 1, ||I(f_n) - I(f_m)||_{L^2(d\mathbb{P})} \to 0, \text{ as } n, m \to \infty.$$

But we have already done a calculation which confirms this: When we noted that the stochastic integral is continuous in the sense that if $g, h \in \mathcal{H}_0^2$ are close then $I(g)$ and $I(h)$ are close. Here we simply use $g = f_n - f_m$ and $h \equiv 0$ to get that $I(f_n - f_m)$ and $I(0) = 0$ are close. To be a bit more specific: As $n, m \to \infty$ we have

$$||I(f_n) - I(f_m)||_{L^2(d\mathbb{P})} = ||I(f_n - f_m)||_{L^2(d\mathbb{P})} = ||f_n - f_m||_{L^2(d\mathbb{P}\times dt)} \to 0.$$

The reason we are interested in the fact that $\{I(f_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(d\mathbb{P})$ is because the space $L^2(d\mathbb{P})$ is *complete*, i.e., *any* Cauchy sequence in it converges to a limit, and that limit remains in the space $L^2(d\mathbb{P})$.

So, we have a Cauchy sequence $\{I(f_n)\}_{n=1}^{\infty}$ of random variables in $L^2(d\mathbb{P})$ and we know that every Cauchy sequence in $L^2(d\mathbb{P})$ has a limit in $L^2(d\mathbb{P})$. Then *our* Cauchy sequence has a limit in $L^2(d\mathbb{P})$ and we *denote* this limit by $I(f)$. Thus for any stochastic process $f \in \mathcal{H}^2$ we have shown that there exists a corresponding object $I(f)$ in $L^2(d\mathbb{P})$ which we call *the stochastic integral of f with respect to Brownian motion.*

The correspondence between f and $I(f)$ was set up through a sequence $\{f_n\}_{n=1}^{\infty}$ of processes in $\mathcal{H}_0^2$. A natural question is: If we had a different sequence, $\{g_n\}_{n=1}^{\infty}$, in $\mathcal{H}_0^2$, would we still get the same result that f corresponds to $I(f)$, or would f correspond to some other object in $L^2(d\mathbb{P})$? The answer to this question is NO!, the object which we have denoted $I(f)$ is unique for f. To prove this, let $\{g_n\}_{n=1}^{\infty}$ be another sequence in $\mathcal{H}_0^2$ which approximates f. Then, by the *triangle inequality* [1] we have

$$||f_n - g_n||_{L^2(d\mathbb{P}\times dt)} \leqslant ||f_n - f||_{L^2(d\mathbb{P}\times dt)} + ||g_n - f||_{L^2(d\mathbb{P}\times dt)} \to 0, \text{ as } n \to \infty.$$

Because the stochastic process $\{f_n - g_n\}_{n=1}^{\infty}$ are elements of $\mathcal{H}_0^2$ we may apply the Itô isometry to get

$$||I(f_n) - I(g_n)||_{L^2(d\mathbb{P})} = ||I(f_n - g_n)||_{L^2(d\mathbb{P})} = ||f_n - g_n||_{L^2(d\mathbb{P}\times dt)} \to 0.$$

This is what we need to show that $I(g_n)$ converges to the same limit, $I(f)$, as $I(f_n)$ because

$$||I(g_n) - I(f)||_{L^2(d\mathbb{P})} \leqslant ||I(g_n) - I(f_n)||_{L^2(d\mathbb{P})} + ||I(f_n) - I(f)||_{L^2(d\mathbb{P})} \to 0,$$

since we have already shown that $I(f_n)$ converges to $I(f)$.

Thus we have established that for every stochastic process $f \in \mathcal{H}^2$ there exists a unique random variable $I(f)$ in $L^2(d\mathbb{P})$, which we call *the stochastic integral of f with respect to Brownian motion.*

[1] The triangle inequality for stochastic processes $x, y \in L^2(d\mathbb{P} \times dt)$ reads:

$$||x - y||_{L^2(d\mathbb{P}\times dt)} \leqslant ||x||_{L^2(d\mathbb{P}\times dt)} + ||y||_{L^2(d\mathbb{P}\times dt)}.$$

Note 1 (The issue of uniqueness in $L^2(d\mathbb{P})$). *Uniqueness in $L^2(d\mathbb{P})$ is quite a vague concept. The reason is that any two random variables $X, Y \in L^2(d\mathbb{P})$ are said to be equal if $||X - Y||_{L^2(d\mathbb{P})} = 0$. This is the same as stating that*

$$\int_\Omega |X(\omega) - Y(\omega)|\, d\mathbb{P}(\omega) = 0.$$

However, this does not *imply that $X(\omega) = Y(\omega)$ for every $\omega \in \Omega$ (as we would expect to hold, intuitively, if X and Y were to be equal); It only implies that $X(\omega) = Y(\omega)$ for every $\omega \in A$, where $A \subseteq \Omega$ is a subset of Ω such that $\mathbb{P}\{\Omega \setminus A\} = 0$.*

As an illustrative example of just how vague the concept of uniqueness in $L^2(d\mathbb{P})$ is, consider the case when the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is such that $\Omega = [0,1]$, $\mathbb{P}(A) = Leb(A)$ (The Lebesgue measure on $[0,1]$; A generalisation of the concept of length) and $\mathcal{F}$=The Borel sigma-algebra on $[0,1]$. Define the random variables X and Y on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$X(\omega) \equiv \begin{cases} -1, & \omega \in [0,1] \setminus A; \\ 1, & \omega \in A, \end{cases}$$

and

$$Y(\omega) = 1, \omega \in [0,1],$$

where $A = [0,1] \setminus ([0,1] \cap \mathbb{Q})$ is the set of irrational numbers in [0,1]. Then

$$\begin{aligned} &\int_\Omega |X(\omega) - Y(\omega)|^2\, d\mathbb{P}(\omega) = \int_A |X(\omega) - Y(\omega)|^2\, d\mathbb{P}(\omega) \\ &+ \int_{[0,1]\setminus A} |X(\omega) - Y(\omega)|^2\, d\mathbb{P}(\omega) = \int_A |1-1|^2\, d\mathbb{P}(\omega) + \int_{[0,1]\setminus A} |-1-1|^2\, d\mathbb{P}(\omega) \\ &= 4\int_{[0,1]\setminus A} d\mathbb{P}(\omega) = 4\mathbb{P}\{[0,1] \setminus A\} = 4\mathbb{P}\{[0,1] \cap \mathbb{Q}\} = 0. \end{aligned}$$

The probability $\mathbb{P}\{[0,1] \cap \mathbb{Q}\}$ is zero because $[0,1] \cap \mathbb{Q}$ is the set of rational numbers in $[0,1]$ and $\mathbb{P}\{[0,1] \cap \mathbb{Q}\}$ is the length of the set of rational numbers of $[0,1]$, which is zero.

Thus we see that $\int_\Omega |X(\omega) - Y(\omega)|^2\, d\mathbb{P}(\omega) = 0$, which, in the space $L^2(d\mathbb{P})$, is the same thing as saying that the random variables

$$X(\omega) \equiv \begin{cases} -1, & \omega \in [0,1] \setminus A; \\ 1, & \omega \in A, \end{cases}$$

and

$$Y(\omega) = 1, \omega \in [0,1],$$

are equal.

The price we have paid for not being able to construct a stochastic integral in the usual way by means of Riemann-Stieltjes integrals is that the stochastic integral, $I(f)$, for $f \in \mathcal{H}^2$ is a very fuzzy object, in the sense that we cannot represent it geometrically as some area under the graph of the function f. If we insist on retaining a geometric representation of $I(f)$ as the area under graph of

f, then we would have a whole bunch of different areas representing $I(f)$ and no means to tell then apart, due to the problem of uniqueness in $L^2(d\mathbb{P})$ discussed above.

Recall that the reason we cannot construct a stochastic integral in the usual way by means of Riemann-Stieltjes integrals, is the fact that Brownian motion has positive *quadratic variation*, i.e.,

$$\mathbb{P}\{\text{For every } t \in [0,\infty), [B,B]_t = t\} = 1.$$

Ito isometry

Lemma 3 (Ito isometry in $\mathcal{H}^2$). *Let $f \in \mathcal{H}^2$ be any stochastic process. Then*

$$||I(f)||_{L^2(d\mathbb{P})} = ||f||_{L^2(d\mathbb{P}\times dt)}.$$

Proof. Let $f \in \mathcal{H}^2$ be any stochastic process. Then by the Approximation Theorem there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of processes $f_n \in \mathcal{H}_0^2$ such that

$$||f - f_n||_{L^2(d\mathbb{P}\times dt)} \to 0, \text{ as } n \to \infty.$$

This implies that $||f_n||_{L^2(d\mathbb{P}\times dt)} \to ||f||_{L^2(d\mathbb{P}\times dt)}$, because

$$\big|\, ||f_n||_{L^2(d\mathbb{P}\times dt)} - ||f||_{L^2(d\mathbb{P}\times dt)} \big| \leqslant ||f_n - f||_{L^2(d\mathbb{P}\times dt)} \to 0.$$

We have also seen that

$$||I(f_n) - I(f)||_{L^2(d\mathbb{P})} \to 0,$$

which, by the same reason as for f_n and f, implies that

$$||I(f_n)||_{L^2(d\mathbb{P})} \to ||I(f)||_{L^2(d\mathbb{P})}.$$

But, since $f_n \in \mathcal{H}_0^2$ we may apply the Itô isometry for processes in $\mathcal{H}_0^2$ to observe that

$$||I(f_n)||_{L^2(d\mathbb{P})} = ||f_n||_{L^2(d\mathbb{P}\times dt)}.$$

Thus we have obtained the results that

$$||f_n||_{L^2(d\mathbb{P}\times dt)} \to ||f||_{L^2(d\mathbb{P}\times dt)}$$

and

$$||f_n||_{L^2(d\mathbb{P}\times dt)} \to ||I(f)||_{L^2(d\mathbb{P})}.$$

Since there can be only one limit of the sequence $\{||f_n||\}_{n=1}^{\infty}$ of positive real numbers, we are forced to conclude that

$$||I(f)||_{L^2(d\mathbb{P})} = ||f||_{L^2(d\mathbb{P}\times dt)}.$$

This proves the Itô isometry for processes in $\mathcal{H}^2$. $\square$

Let us familiarise ourselves with the results we have obtained so far.

Consider the one dimensional Brownian motion $B = \{B_t\}_{t\in[0,T]}$. Then $B \in \mathcal{H}^2$, because the defining properties of the space $\mathcal{H}^2$ are satisfied. (The first property, that the map $(\omega, t) \mapsto B(\omega, t)$ is $\mathcal{F}_T \times \mathcal{B}_{[0,T]}$-measurable, might be somewhat difficult to prove.) Then the stochastic integral $I(B)$, which we denote by

$$\int_0^T B_s \, dB_s,$$

exists as a unique random variable of $L^2(d\mathbb{P})$. By the Itô isometry for processes in $\mathcal{H}^2$ we may compute the expectation

$$\mathbb{E}\left\{ \left| \int_0^T B_t \, dB_t \right|^2 \right\} = ||I(B)||_{L^2(d\mathbb{P})} = ||B||_{L^2(d\mathbb{P}\times dt)}$$
$$= \int_0^T \mathbb{E}\{|B_t|^2\} \, dt = \int_0^T t \, dt = \frac{T}{2}.$$

Next we take as (stochastic) process the collection $\{t\}_{t\in[0,T]}$, which also is an element of the space $\mathcal{H}^2$. Then we know that the stochastic integral

$$I(t) = \int_0^T t \, dB_t$$

exists as a unique random variable in $L^2(d\mathbb{P})$ and by the Itô isometry we compute the expected value

$$\mathbb{E}\left\{ \left| \int_0^T t \, dB_t \right|^2 \right\} = ||I(t)||_{L^2(d\mathbb{P})} = ||t||_{L^2(d\mathbb{P}\times dt)}$$
$$= \int_0^T \mathbb{E}\{|t|^2\} \, dt = \int_0^T t^2 \, dt = \frac{T^3}{3}.$$

As a generalisation of the stochastic integral $\int_0^T B_t \, dB_t$ we may consider taking the process $B^n = \{B_t^n\}_{t\in[0,T]}$, for any positive integer $n \in \mathbb{N}$. Then $B^n \in \mathcal{H}^2$ and we therefore know that the stochastic integral $\int_0^T B_t^n \, dB_t$ exists as a unique random variable in $L^2(d\mathbb{P})$. Since any linear combination

$$\sum_{k=1}^m a_k \int_0^T B_t^k \, dB_t$$

of random variables $\int_0^T B_t^k \, dB_t \in L^2(d\mathbb{P})$ is again a random variable in $L^2(d\mathbb{P})$, we see that

$$\sum_{k=1}^m a_k \int_0^T B_t^k \, dB_t \in L^2(d\mathbb{P}).$$

But we also know that the stochastic integral is *linear* in the sense that

$$\int_0^T f_t \, dB_t + \int_0^T g_t \, dB_t = \int_0^T (f_t + g_t) \, dB_t, \text{ for any processes } f, g \in \mathcal{H}^2.$$

Thus we have the result that for any *finite* linear combination, $\sum_{k=1}^m a_k B^k$, of Brownian motions, B^k,

$$\int_0^T \left\{ \sum_{k=1}^m a_k B_t^k \right\} dB_t \in L^2(d\mathbb{P}).$$