Solving stochastic differential equations

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Consider a stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= a(t, X_t)\, dt + b(t, X_t)\, dB_t; \\ X_0 &= x_0. \end{aligned} \tag{1}$$

If we are interested in finding the *strong* solution to this equation then we are searching for a function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that $X_t = f(t, B_t)$. This is so because the way X_t changes is governed by t and B_t; The coefficients $a(t, X_t)$ and $b(t, X_t)$ only describe the effects changes in t and B_t respectively have on changes in X_t.

Because the process $\{X_t\}$ has the dynamics as described in (1), there will be a corresponding dynamics for the process $ft, B_t)$. The dynamics is obtained by applying the Itô Formula to $f(t, B_t)$.

This gives

$$\begin{aligned} f(t, B_t) = f(0,0) &+ \int_0^t \left\{ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, B_s) + \frac{\partial f}{\partial s}(s, B_s) \right\} ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s. \end{aligned} \tag{2}$$

If we compare this with the dynamics for X_t

$$X_t = x_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s)\, dB_s. \tag{3}$$

If we choose the function f so that it satisfies the following system of partial differential equations then we will have a *candidate* for the solution of SDE (1):

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) &= a(s, f(s, x)); \\ \frac{\partial f}{\partial x}(s, x) &= b(s, f(s, x)); \\ f(0, 0) &= x_0. \end{aligned} \tag{4}$$

This technique is useful mostly for *linear* coefficient functions $a(s, x)$ and $b(s, x)$.

Example 1. *Consider the SDE*

$$\begin{aligned} dX_t &= dt + dB_t; \\ X_0 &= x_0, \end{aligned}$$

which is the simplest linear SDE imaginable. We know that the strong solution to this equation is $X_t = x_0 + t + B_t$. Let us see what the coefficient matching technique gives us. The system we need to solve in this case is

$$\begin{aligned} \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) &= 1; \\ \frac{\partial f}{\partial x}(s,x) &= 1; \\ f(0,0) &= x_0. \end{aligned} \tag{5}$$

The solution is computed as follows.

$$\begin{aligned} \frac{\partial f}{\partial x}(s,x) = 1 &\Rightarrow f(s,x) = x + g(s); \\ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = 1 &\Rightarrow g'(s) = 1 \Rightarrow g(s) = s + c_0; \\ f(0,0) = x_0 &\Rightarrow c_0 = x_0 \end{aligned}$$

Thus we have obtained $f(s,x) = x_0 + t + x$, and the candidate strong solution to the SDE is $f(t, B_t) = x_0 + t + B_t$, which in this case actually is *the strong solution to* (5)*.*

Example 2. *Consider the so called Ornstein-Uhlenbeck SDE*

$$\begin{aligned} dX_t &= -rX_t dt + \sigma dB_t; \\ X_0 &= x_0, \end{aligned}$$

where $r, \sigma \in \mathbb{R}$ are constants. If we apply the coefficient matching technique we get the system

$$\begin{aligned} \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) &= -rf(t,x); \\ \frac{\partial f}{\partial x}(s,x) &= \sigma; \\ f(0,0) &= x_0. \end{aligned}$$

This results in an equation $f(t,x) = \sigma x + g(t)$, where the function g satisfies the ODE

$$rg(t) + g'(t) = -r\sigma x. \tag{6}$$

Since the right-hand side of (6) *is a function of t alone, and the left-hand side a function of x alone, these have to be constant. So we are forced to conclude that x is constant. But this is absurd, since x is a variable quantity.*

This simple example shows that even for harmless looking linear coefficient functions $a(t,x)$ and $b(t,x)$, the coefficient matching technique might lead to nonsense!