Solving stochastic differential equations

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July 12, 2005

Consider a stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= a(t, X_t)\, dt + b(t, X_t)\, dB_t; \\ X_0 &= x_0. \end{aligned} \tag{1}$$

If we are interested in finding the *strong* solution to this equation then we are searching for a function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that $X_t = f(t, B_t)$. This is so because the way X_t changes is governed by t and B_t; The coefficients $a(t, X_t)$ and $b(t, X_t)$ only describe the effects changes in t and B_t respectively have on changes in X_t.

Because the process $\{X_t\}$ has the dynamics as described in (1), there will be a corresponding dynamics for the function f. These changes are obtained by applying the Ito Formula to $f(t, B_t)$.

This gives

$$\begin{aligned} f(t, B_t) = f(0,0) &+ \int_0^t \left\{ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, B_s) + \frac{\partial f}{\partial s}(s, B_s) \right\} ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s. \end{aligned} \tag{2}$$

If we compare this with the dynamics for X_t

$$X_t = x_0 + \int_0^t a(s, X_s)\, ds + \int_0^t b(s, X_s)\, dB_s. \tag{3}$$

If we choose the function f so that it satisfies the following system of partial differential equations then we will have a *candidate* for the solution of SDE (1):

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, x) + \frac{\partial f}{\partial s}(s, x) &= a(s, f(s, x)); \\ \frac{\partial f}{\partial x}(s, x) &= b(s, f(s, x)); \\ f(0,0) &= x_0. \end{aligned} \tag{4}$$

This technique is useful mostly for *linear* coefficient functions $a(s, x)$ and $b(s, x)$.

Example 1. *Consider the SDE*

$$\begin{aligned} dX_t &= dt + dB_t; \\ X_0 &= x_0, \end{aligned}$$

which is the simplest linear SDE imaginable. We know that the strong solution to this equation is $X_t = x_0 + t + B_t$. Let us see what the coefficient matching technique gives us. The system we need to solve in this case is

$$\begin{aligned} \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) &= 1; \\ \frac{\partial f}{\partial x}(s,x) &= 1; \\ f(0,0) &= x_0. \end{aligned} \tag{5}$$

The solution is computed as follows.

$$\begin{aligned} \frac{\partial f}{\partial x}(s,x) = 1 &\Rightarrow f(s,x) = x + g(s); \\ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = 1 &\Rightarrow g'(s) = 1 \Rightarrow g(s) = s + c_0; \\ f(0,0) = x_0 &\Rightarrow c_0 = x_0 \end{aligned}$$

Thus we have obtained $f(s,x) = x_0 + t + x$, and the candidate strong solution to the SDE is $f(t, B_t) = x_0 + t + B_t$, which in this case actually is *the strong solution to* (5)*.*

Example 2. *Consider the so called Ornstein-Uhlenbeck SDE*

$$\begin{aligned} dX_t &= -rX_t dt + \sigma dB_t; \\ X_0 &= x_0, \end{aligned}$$

where $r, \sigma \in \mathbb{R}$ are constants. If we apply the coefficient matching technique we get the system

$$\begin{aligned} \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) &= -rf(t,x); \\ \frac{\partial f}{\partial x}(s,x) &= \sigma; \\ f(0,0) &= x_0. \end{aligned}$$

This results in an equation $f(t,x) = \sigma x + g(t)$, where the function g satisfies the ODE

$$rg(t) + g'(t) = -r\sigma x. \tag{6}$$

Since the right-hand side of (6) *is a function of t alone, and the left-hand side a function of x alone, these have to be constant. So we are forced to conclude that x is constant. But this is absurd, since x is a variable quantity.*

This simple example shows that even for harmless looking linear coefficient functions $a(t,x)$ and $b(t,x)$, the coefficient matching technique might lead to nonsense!

1 Techniques based on coefficient matching

Consider the SDE

$$\begin{aligned} dX_t &= b(t, X_t)\, dB_t; \\ X_0 &= x_0. \end{aligned} \tag{7}$$

We want to find a transformation $f(t,x)$ such that $f(t,X_t)=B_t$.

Consider a fixed $t\in[0,T]$ and define the map $g_t : x\mapsto f(t,x)$. Then $g_t(X_t)=B_t$ and if g_t is *injective*, i.e., if it has an inverse map, g_t^{-1}, then $X_t=g_t^{-1}(B_t)$, and the solution to the SDE (7) is obtained as the process $\{g_t^{-1}(B_t)\}_{t\in[0,T]}$.

The Itô Formula applied to the process $f(t,X_t)$ gives

$$\begin{aligned}f(t,X_t)=f(0,x_0)&+\int_0^t\left\{\frac{\partial f}{\partial s}(s,X_s)+\frac{1}{2}b^2(s,X_s)\frac{\partial^2 f}{\partial x^2}(s,X_s)\right\}ds\\&+\int_0^t b(s,X_s)\frac{\partial f}{\partial x}(s,X_s)\,dB_s.\end{aligned}$$

Let the function $(t,x)\mapsto f(t,x)$ satisfy the following system of partial differential equations:

$$\begin{aligned}\frac{\partial f}{\partial t}(t,x)+\frac{1}{2}b^2(t,x)\frac{\partial^2 f}{\partial x^2}(t,x)&=0;\\ b(t,x)\frac{\partial f}{\partial x}(t,x)&=1;\\ f(0,x_0)&=0.\end{aligned}$$

If such a function exists then the Itô Formula tells us that $f(t,X_t)=B_t$.

2 Calculation of mean values

Let X solve the SDE

$$\begin{aligned}dX_t&=a(X_t)\,dt+b(X_t)\,dB_t;\\ X_0&=x_0.\end{aligned}\tag{8}$$

By the general Itô Formula for processes like X we have

$$\begin{aligned}f(X_t)=f(x_0)&+\int_0^t a(X_s)f'(X_s)\,ds+\int_0^t b(X_s)f'(X_s)\,dB_s\\&+\frac{1}{2}\int_0^t b^2(X_s)f''(X_s)\,ds.\end{aligned}\tag{9}$$

Guided by Hand-in 3, Problem 3.2, we ask the question: *For which functions f do we get an integral equation*

$$\mathbb{E}\{f(X_t)\}=f(x_0)+c\int_0^t\mathbb{E}\{f(X_s)\}\,ds,$$

for the mean $\mathbb{E}\{f(X_t)\}$? By taking expectations in the Ito Formula we get

$$\mathbb{E}\{f(X_t)\}=f(x_0)+\int_0^t\mathbb{E}\Big\{a(X_s)f'(X_s)+\frac{1}{2}b^2(X_s)f''(X_s)\Big\}\,ds.$$

From this we note that if the function f solves the differential equation

$$a(x)f'(x)+\frac{1}{2}b^2(x)f''(x)=cf(x),$$

for some constant $c \in \mathbb{R}$, then we get the desired integral equation.

Thus we have obtained the result that if X solves the SDE

$$\begin{aligned} dX_t &= a(X_t)\,dt + b(X_t)\,dB_t; \\ X_0 &= x_0, \end{aligned}$$

and f solves the ODE

$$a(x)f'(x) + \frac{1}{2}b^2(x)f''(x) = cf(x),$$

then

$$\mathbb{E}\{f(X_t)\} = f(x_0)e^{ct}.$$

Example 3. *If $a(x) = rx$ and $b(x) = \sigma x$ for some constants $r, \sigma \in \mathbb{R}$, then every monomial $f(x) = x^n$ solves the ODE*

$$rxf'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) = c_n f(x),$$

where the constant, c_n, is given[1] *by $c_n = nr + \frac{n(n-1)}{2}\sigma^2$. The corresponding SDE is the usual geometric Brownian motion*

$$dX_t = rX_t\,dt + \sigma X_t\,dB_t, \quad X_0 = x_0.$$

Consequently we have the result that

$$\mathbb{E}\{X_t^n\} = x_0^n e^{(nr+n(n-1)\sigma^2/2)t},$$

special cases of which are

$$\mathbb{E}\{X_t\} = x_0 e^{rt}$$

and

$$\mathbb{E}\{X_t^2\} = x_0^2 e^{(2r+\sigma^2)t}.$$

Thus we have obtained

$$Var[X_t] = \mathbb{E}\{X_t^2\} - (\mathbb{E}\{X_t\})^2 = x_0^2 e^{2rt}(e^{\sigma^2 t} - 1).$$

If the coefficients r and σ satisfy $2r + \sigma^2 = 0$, i.e. if we have the SDE

$$dX_t = -\sigma^2/2X_t\,dt + \sigma X_t\,dB_t, \quad X_0 = x_0,$$

then we get

$$Var[X_t] = \mathbb{E}\{X_t^2\} - (\mathbb{E}\{X_t\})^2 = x_0^2(1 - e^{-\sigma^2 t}) \uparrow x_0^2, \text{ as } t \to \infty.$$

Let

$$p(x) = \sum_{k=1}^{n} a_k x^k$$

be any polynomial with coefficients $c_k \in \mathbb{R}$. Then, by the linearity of expectation,

$$\mathbb{E}\{p(X_t)\} = \sum_{k=1}^{n} a_k x_0^k e^{(kr+\frac{k(k-1)}{2}\sigma^2)t}.$$

[1] Just insert the monomial $f(x) = x^n$ into the ODE and equate the coefficient of x^n on the left-hand side $(nr + \frac{n(n-1)}{2}\sigma^2)$ with that of x^n on the right-hand side.

Since one is often interested in computing the n^{th} moment, $\mathbb{E}\{X_t^n\}$, of the solution X_t governed by an SDE, it is interesting to know for which coefficient functions $a(x)$ and $b(x)$ this method can be used to compute $\mathbb{E}\{X_t^n\}$, i.e., we ask the question: *For which coefficient functions $a(x)$ and $b(x)$ are the monomials $f(x) = x^n$ solutions to the ODE*

$$a(x)f'(x) + \frac{1}{2}b^2(x)f''(x) = cf(x)?$$

To answer this question, we simply assume that any monomial is a solution to the ODE and see what this implies regarding the coefficients a(x) and b(x). We get the equation

$$n\frac{a(x)}{x} + \frac{n(n-1)}{2}\left\{\frac{b(x)}{x}\right\}^2 = c,$$

where $c \in \mathbb{R}$ is a constant. In order for this equation to make sense, $a(x) = a_0 x$ and $b(x) = b_0 x$, for some constants $a_0, b_0 \in \mathbb{R}$.

Thus we have come to the conclusion that this method of computing the moments $\mathbb{E}\{X_t^n\}$ works *only* for the geometric Brownian motion.

Note 1. *The ODE-SDE connection is useful for SDEs other that geometric Brownian motion. We have only shown that the connection is not useful if we are interested in finding the expectations $\mathbb{E}\{X_t^n\}$. If we are interested in computing the expectation $\mathbb{E}\{f(X_t)\}$, where f satisfies the ODE, then the connection is useful.*

3 Reducible equations

Consider solving the SDE

$$\begin{aligned} dX_t &= a(t, X_t)\,dt + b(t, X_t)\,dB_t \\ X_0 &= x_0. \end{aligned}$$

Suppose you can find a transformation $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ such that $Y_t \equiv f(t, X_t)$ is governed by the SDE

$$\begin{aligned} dY_t &= r(t)\,dt + \sigma(t)\,dB_t \\ Y_0 &= f(0, x_0) = 0. \end{aligned} \tag{10}$$

What kind of coefficient functions a and b admit such a transformation? In what way are the x-independent coefficients r and σ determined by the coefficients a and b? We will now attempt to find answers to these questions.

By the Itô Formula,

$$dY_t = \{\partial_t f + a\partial_x f + \frac{1}{2}b^2\partial_{xx}^2 f\}\,dt + \{b\partial_x f\}\,dB_t,$$

where we have suppressed most of the notation in favour of brevity; E.g. $a\partial_x f$ reads $\{a(t,x)\frac{\partial}{\partial x}f(t,x)\}|_{x=X_t}$. Our demands (10) on the transformation f give the system of PDE

$$\begin{aligned} r(t) &= \partial_t f + \sigma(t)\left\{\frac{a}{b} - \frac{1}{2}\partial_x b\right\}; \\ \partial_x f &= \frac{\sigma(t)}{b}. \end{aligned}$$

The fact that $r(t)$ does not depend on x gives

$$\partial_x r(t) = 0 = \partial_{xt}^2 f + \sigma(t)\partial_x\Big\{\frac{a}{b} - \frac{1}{2}\partial_x b\Big\} \Leftrightarrow \partial_{xt}^2 f = -\sigma(t)\partial_x\Big\{\frac{a}{b} - \frac{1}{2}\partial_x b\Big\}.$$

Applying the operator ∂_t on the equation $\partial_x f = \frac{\sigma(t)}{b}$ gives

$$\partial_{tx}^2 f = \frac{1}{b^2}(b\partial_t\sigma(t) - \sigma(t)\partial_t b).$$

If the function f is assumed to have continuous partial second derivatives, then we may equate the mixed partial derivatives $\partial_{tx}^2 f$ and $\partial_{xt}^2 f$ to obtain a single equation

$$b\partial_t\sigma(t) - \sigma(t)\partial_t b = -b^2\sigma(t)\partial_x\Big\{\frac{a}{b} - \frac{1}{2}\partial_x b\Big\}.$$

Assuming that $\sigma(t)b(t,x) \neq 0$ for no pair $(t,x) \in [0,T] \times \mathbb{R}$, we may divide the equation by $\sigma(t)b(t,x)$ to get

$$\frac{\partial_t\sigma(t)}{\sigma(t)} - \frac{\partial_t b}{b} = -b\partial_x\Big\{\frac{a}{b} - \frac{1}{2}\partial_x b\Big\}.$$

Since the ratio $\frac{\partial_t\sigma(t)}{\sigma(t)}$ does not depend on the variable x, if we apply the operator ∂_x to the equation, we arrive at a PDE determining for what kind of coefficient functions a and b the SDE $dX_t = a(t,X_t)\,dt + b(t,X_t)\,dB_t$ can be transformed to an SDE $dY_t = r(t)\,dt + \sigma(t)\,dB_t$:

$$\partial_x\Big\{\frac{\partial_t b}{b} - b\partial_x\Big\{\frac{a}{b} - \frac{1}{2}\partial_x b\Big\}\Big\}.$$

Theorem 1 (Reducible SDE). *If the coefficient functions $a(t,x)$ and $b(t,x)$ satisfy the partial differential equation*

$$\frac{\partial}{\partial x}\Big\{\frac{\frac{\partial b}{\partial t}(t,x)}{b(t,x)} - b(t,x)\frac{\partial}{\partial x}\Big\{\frac{a(t,x)}{b(t,x)} - \frac{1}{2}\frac{\partial b}{\partial x}(t,x)\Big\}\Big\} = 0,$$

then there exists a transformation, $Y_t = f(t,X_t)$, transforming the stochastic differential equation

$$\begin{aligned} dX_t &= a(t,X_t)\,dt + b(t,X_t)\,dB_t \\ X_0 &= x_0, \end{aligned}$$

to the stochastic differential equation

$$\begin{aligned} dY_t &= r(t)\,dt + \sigma(t)\,dB_t \\ Y_0 &= f(0,x_0) = 0. \end{aligned}$$

The coefficients, $r(t)$ and $\sigma(t)$, of the transformed equation are determined by the system of partial differential equations

$$\begin{aligned} \frac{d\sigma(t)}{dt} &= \sigma(t)\Big\{\frac{\frac{\partial b}{\partial t}(t,x)}{b(t,x)} - b(t,x)\frac{\partial}{\partial x}\Big\{\frac{a(t,x)}{b(t,x)} - \frac{1}{2}\frac{\partial b}{\partial x}(t,x)\Big\}\Big\} \\ r(t) &= \sigma(t)\Big\{\frac{a(t,x)}{b(t,x)} - \frac{1}{2}\frac{\partial b}{\partial x}(t,x)\Big\} + \frac{\partial}{\partial t}\Big\{\sigma(t)\int_{x_0}^{x}\frac{dy}{b(t,y)}\Big\}, \end{aligned}$$

and the transformation, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, by the system of partial differential equations

$$\frac{\partial f}{\partial x}(t,x) = \frac{\sigma(t)}{b(t,x)}$$
$$\frac{\partial f}{\partial t}(t,x) = r(t) - \sigma(t)\left\{\frac{a(t,x)}{b(t,x)} - \frac{1}{2}\frac{\partial b}{\partial x}(t,x)\right\}.$$

Example 4 (Geometric Brownian motion). *Consider the geometric Brownian motion governed by the stochastic differential equation*

$$dX_t = a_0 X_t\,dt + b_0 X_t\,dB_t$$
$$X_0 = x_0.$$

We first investigate whether it is possible to transform this equation to an equation of the form

$$dY_t = r(t)\,dt + \sigma(t)\,dB_t$$
$$Y_0 = f(0,x_0) = 0.$$

In order for this to be possible, the coefficient functions $a(t,x) = a_0 x$ and $b(t,x) = b_0 x$ have to satisfy the partial differential equation

$$\frac{\partial}{\partial x}\left\{\frac{\frac{\partial b}{\partial t}(t,x)}{b(t,x)} - b(t,x)\frac{\partial}{\partial x}\left\{\frac{a(t,x)}{b(t,x)} - \frac{1}{2}\frac{\partial b}{\partial x}(t,x)\right\}\right\} = 0.$$

A simple computation shows that this is indeed the case. We next proceed to determine the coefficients $r(t)$ and $\sigma(t)$ of the transformed equation. The equation for $\sigma(t)$ in this case reads

$$\frac{d\sigma(t)}{dt} = 0,$$

implying that $\sigma(t) = \sigma = constant$. This gives an equation determining the coefficient $r(t)$ to be

$$r(t) = \frac{\sigma}{b_0}\left(a_0 - \frac{b_0^2}{2}\right) \equiv r = constant.$$

Thus, the transformed SDE reads

$$dY_t = r\,dt + \sigma\,dB_t$$
$$Y_0 = f(0,x_0) = 0,$$

whose solution is $Y_t = rt + \sigma B_t$. All that is left for us to find the solution of the original SDE

$$dX_t = a_0 X_t\,dt + b_0 X_t\,dB_t$$
$$X_0 = x_0,$$

is to find the transformation $f(t,x)$ connecting Y_t and X_t through

$$Y_t = f(t,X_t)$$

and then to invert it to get

$$X_t = f^{-1}(t,Y_t),$$

if the inverse can be obtained. The partial differential equations governing the transformation f(t,x) reads

$$\frac{\partial f}{\partial x}(t,x) = \frac{\sigma}{b_0}\frac{1}{x}$$
$$\frac{\partial f}{\partial t}(t,x) = 0,$$

giving $f(t,x) = \frac{\sigma}{b_0}\log(x) + c_0$*, where* c_0 *is some constant, determined by the condition* $0 = f(0,x_0)$*. This gives* $c_0 = -\frac{\sigma}{b_0}\log(x_0)$ *and consequently*

$$f(t,x) = \frac{\sigma}{b_0}\log(\frac{x}{x_0}).$$

The inverse can be obtained and reads

$$f^{-1}(t,x) = x_0 e^{\frac{b_0}{\sigma}x}.$$

If we insert the transformed process $Y_t = rt + \sigma B_t$ *we get*

$$X_t = f^{-1}(t,Y_t) = x_0 e^{\frac{b_0}{\sigma}(rt+\sigma B_t)}.$$

Now,

$$\frac{b_0}{\sigma}r = \frac{b_0}{\sigma}\frac{\sigma}{b_0}\left(a_0 - \frac{b_0^2}{2}\right) = a_0 - \frac{b_0^2}{2},$$

resulting in the geometric Brownian motion

$$X_t = x_0 e^{(a_0 - \frac{b_0^2}{2})t + b_0 B_t},$$

as the solution to the stochastic differential equation

$$dX_t = a_0 X_t\,dt + b_0 X_t\,dB_t$$
$$X_0 = x_0.$$

The reason the procedure of finding the solution of the SDE governing the geometric Brownian motion was so long-winded, is that we used a general technique finding it. It is almost invariably the case that whenever a general result is applied to a specific problem, lengthy calculations are the result.

We have seen that for this problem, the technique of coefficient matching offered a much faster route towards finding the geometric Brownian motion. However, we have also seen that the technique of coefficient matching is far from perfect. Indeed it can fail for simple linear stochastic differential equations such as the one governing the Ornstein-Uhlenbeck process.

Example 5. *Consider the Ornstein-Uhlenbeck process*

$$\begin{aligned} dX_t &= -rX_t\,dt + \sigma\,dB_t; \\ X_0 &= x_0, \end{aligned} \tag{11}$$

for which the technique of coefficient matching failed to provide a solution. Let us apply the technique of reduction and see if this works. There exists a reducing transformation, $f(t,X_t)$*, if the coefficient functions of the Ornstein-Uhlenbeck process,* $a(t,x) = -rx$ *and* $b(t,x) = \sigma$*, satisfy the partial differential equation*

$$\partial_x\left\{\frac{\partial_t b}{b} - b\partial_x\left\{\frac{a}{b} - \frac{1}{2}\partial_x b\right\}\right\} = 0. \tag{12}$$

A simple computation shows that this is indeed the case. Thus there exists a transformation $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *such that the process* $Y_t = f(t, X_t)$ *is governed by the SDE*

$$\begin{aligned} dY_t &= \alpha(t)\, dt + \beta(t)\, dB_t; \\ Y_0 &= f(0, x_0), \end{aligned} \tag{13}$$

where the coefficients $\alpha(t)$ *and* $\beta(t)$ *are given by*

$$\begin{aligned} \beta'(t) &= \beta(t)\left\{ \frac{\partial_t b}{b} - b\partial_x \left\{ \frac{a}{b} - \frac{1}{2}\partial_x b \right\} \right\}; \\ \alpha(t) &= \beta(t)\left\{ \frac{a}{b} - \frac{1}{2}\partial_x b \right\} + \partial_t \left\{ \beta(t) \int_{x_0}^{x} \frac{dy}{b(t,y)} \right\}. \end{aligned} \tag{14}$$

For us this amounts to

$$\begin{aligned} \beta'(t) &= r\beta(t); \\ \alpha(t) &= -\frac{r}{\sigma} x \beta(t) + \frac{x - x_0}{\sigma} \beta'(t), \end{aligned} \tag{15}$$

i.e.,

$$\begin{aligned} \beta(t) &= b_0 e^{rt}; \\ \alpha(t) &= -\frac{x_0 r}{\sigma} \beta(t), \end{aligned} \tag{16}$$

for some constant $b_0 \in \mathbb{R}$*. The SDE for the process* Y_t *reads*

$$\begin{aligned} dY_t &= -\frac{x_0 r}{\sigma} b_0 e^{rt}\, dt + b_0 e^{rt}\, dB_t; \\ Y_0 &= f(0, x_0). \end{aligned} \tag{17}$$

Next, we will find a transformation $f(t, X_t)$ *such that* $f(t, X_t) = Y_t$*. Such a reducing transformation satisfies the partial differential equations*

$$\begin{aligned} \partial_t f &= \alpha(t) - \beta(t)\left\{ \frac{a}{b} - \frac{1}{2}\partial_x b \right\} = \frac{(x - x_0) r}{\sigma} \beta(t); \\ \partial_x f &= \frac{\beta(t)}{b}. \end{aligned} \tag{18}$$

Everything that is required to solve for $f(t, x)$ *is known and we obtain*

$$f(t, x) = f(0, x_0) + \frac{(x - x_0)}{\sigma} \beta(t).$$

For a fixed $t \in [0, T]$ *we can invert this transformation to obtain*

$$x = f^{-1}(t, y) = x_0 + \frac{\sigma}{\beta(t)} \{ y - f(0, x_0) \},$$

from which we find

$$\begin{aligned} X_t = f^{-1}(t, Y_t) &= x_0 + \frac{\sigma}{\beta(t)} (Y_t - Y_0) = x_0 + \frac{\sigma}{\beta(t)} \int_0^t dY_s \\ &= x_0 - x_0 r \int_0^t \frac{\beta(s)}{\beta(t)}\, ds + \sigma \int_0^t \frac{\beta(s)}{\beta(t)}\, dB_s = x_0 e^{-rt} + \sigma e^{-rt} \int_0^t e^{rs}\, dB_s. \end{aligned}$$

Since the integrand, e^{rs}, in the stochastic integral $\int_0^t e^{rs}\,dB_s$ of the Ornstein-Uhlenbeck process is non-random, the stochastic integral process $\{\int_0^t e^{rs}\,dB_s\}_{t\in[0,T]}$ is a Gaussian process whose expectation function is

$$m(t) = \mathbb{E}\{\int_0^t e^{rs}\,dB_s\} = 0,$$

because the stochastic integral is a martingale, and whose covariance function is

$$Cov\Big(\int_0^s e^{ru}\,dB_u, \int_0^t e^{rv}\,dB_v\Big) = \mathbb{E}\Big\{\int_0^s e^{ru}\,dB_u \int_0^t e^{rv}\,dB_v\Big\}$$
$$= \int_0^{\min(s,t)} \mathbb{E}\{e^{ru}e^{ru}\}\,du = \int_0^{\min(s,t)} e^{2ru}\,du = \frac{1}{2r}\{e^{2r\min(s,t)} - 1\}.$$

From these facts we deduce that the expectation function of the Ornstein-Uhlenbeck process X is

$$\mathbb{E}\{X_t\} = x_0 e^{-rt}$$

and the covariance function is

$$Cov(X_t, X_s) = Cov\Big(x_0 e^{-rt} + \sigma e^{-rt}\int_0^t e^{rv}\,dB_v, x_0 e^{-rs} + \sigma e^{-rs}\int_0^s e^{rv}\,dB_v\Big)$$
$$= \sigma^2 e^{-r(s+t)} Cov\Big(\int_0^s e^{ru}\,dB_u, \int_0^t e^{rv}\,dB_v\Big)$$
$$= \frac{\sigma^2}{2r} e^{-r(s+t)}\{e^{2r\min(s,t)} - 1\}.$$

The reason we calculate the expectation and covariance functions are that they completely determine the Ornstein-Uhlenbeck process, X, since this process is Gaussian.