Stochastic Calculus Part I - 2007
Hand-in No 1, due September 14.

Let $V_g(t)$ denote the variation of g and $[f,g](t)$ the quadratic covariation on $[0,t]$ of functions f and g.

1 Show that $V_g(t) - g(t)$ is a nondecreasing function of t. (1 point)

2 If g is continous, show that (3 points)

a) $V_g(t) = 0 \Rightarrow [g,g](t) = 0$

b) $[g,g](t) = 0 \not\Rightarrow V_g(t) = 0$

c) $[g,g](t) > 0 \Rightarrow V_g(t) = \infty$

3 Let $f(x) = \cos x$ and $g(x) = \begin{cases} 0, & x < \pi \\ 1, & x \geq \pi \end{cases}$. Calculate the Stiltjes integrals

$$\int_0^{2\pi} f(x)dg(x) \text{ and } \int_0^{2\pi} g(x)df(x).$$

Motivate finite variation where necessary. (2 points)

4 Let X and Y be independent standard normal random variables, and $Z = X+Y$. Calculate $\mathrm{E}[\,Z\,|\,Y=y]$, $\mathrm{E}[\,X\,|\,Y=y]$, $\mathrm{E}[\,XYZ\,|\,Y=y]$ and $\mathrm{E}[\,Y\,|\,Z=z]$. (2 points)

Hand-in No 2, due September 21.

Let B_t denote Brownian motion at time t.

1 Find the conditional distribution of B_s given $B_t = b$, where $0 < s < t$. (2 point)

2 Calculate $P(B_{t^2} + tB_1 \leq 1)$, for all $t > 0$. (2 points)

3 Show that $X_t = tB_{1/t}$, $t > 0$ and $X_0 = 0$ is a Brownian motion. (2 points)

4 Simulate B_t to find $P(B_t < t^{1/3})$ for $t \in (0,1)$ and calculate $P(B_t < t^{1/3})$ exact $\forall t \in (0,1)$. Display the results in a graph. (2 points)

Hand-in No 3, Stochastic Calculus Part I, due October 5.

1 Let $X_t = \sum_{k=1}^{N} \xi_k 1_{(k,k+1]}(t)$, where $\xi_k = 1_{\{B_k - B_{k-1} \geq 0\}}$. Argue that X_t is a simple adopted process. Find $\int_0^T X_t dB_t$, and calculate its mean and variance. (3 points)

2 Let $dX_t = 2B_t dB_t + dt$. Find $\mathrm{E}[X_t]$ and $\mathrm{Var}(X_t)$. (1 point)

3 Solve $dX_t = rX_t dt + \sigma X_t dB_t$. (2 points)
Hint: Consider $f(X_t) = \ln(X_t)$.

On a financial market let r_i be the expected increase over time t[years] for market paper i and let σ_i be the volatility for paper i. Then the SDE displaying the dynamic for paper i is

$$dX_t^i = X_t^i(r_i dt + \sigma_i dB_t^i) \tag{1}$$

(B_t^i is independent of B_t^j, $\forall i \neq j$).

4 Let X_t^1 and X_t^2 be two papers run by (1) with $r_1 = \ln(\frac{11}{10})$, $\sigma_1 = \ln(\frac{10}{9})$ and $r_2 = 0$, $\sigma_2 = \ln(\frac{100}{81})$. Simulate 1000 times, using (1), to find approximate values for $E[X_1^i]$ and ξ that gives $P(X_1^i > \xi) = 0.99$ (i.e. the 99%-quantile). Both papers are worth 100 SEK at starting time. (2 points)
Hint: An estimate of ξ is found by taking the 990-largest sample of the 1000.

Hand-in No 4, Stochastic Calculus Part I, due October 12.

1 Let $X_t = \int_0^t B_s ds + \int_0^t s dB_s$. Find $tB_t^2 d\ln(X_t)$. (1 point)

2 Simulate to find implications that $dB_t^2 \neq 2B_t dB_t$ (which would have been the case if B_t was a C^1-function). (2 point)

3 Show that $M_t = e^{t/2}\sin B_t$ is a martingale. Find the variation of M_t. (2 points)
Hint: Itô's formula.

4 Show that a strong solution exist for the SDE $dX_t = X_t dt + \frac{B_t}{\sqrt{t}} dB_t$, $X_0 = 1$. Find it. (3 points)

Hand-in No 5, Stochastic Calculus Part I, due October 19.

1 Find $\mathrm{E}[X_t]$ and $\mathrm{Var}(X_t)$, when $dX_t = \sqrt{X_t + 1}dB_t$. Itô integrals may be assumed martingales. (2 points)

2 Let

$$dZ_t = \left(2e^{\sqrt{2}B_t} + Z_t\right) dt + \left(2e^{\sqrt{2}B_t} + \sqrt{2}Z_t\right) dB_t.$$

Find Z_t when $Z_0 = 0$. (2 points)

3 Find X_t, when $X_0 = 1$ and

$$dX_t = -\left(\sqrt{1 - X_t^2} + \frac{X_t}{2}\right) dt - \sqrt{1 - X_t^2} dB_t.$$

What can you say about uniqueness? (2 points)

4 The Ornstein-Uhlenbeck and Black-Schols processes are defined via the SDE's

$$dX_t = -\alpha X_t dt + \sigma dB_t, \quad \alpha > 0,$$

and

$$dX_t = X_t(r dt + \sigma dB_t),$$

respectively. Show that their (strong) solutions are unique. (2 points)

Hand-in No 6, Stochastic Calculus Part I, due October 26.

1 Let $Y_t = B_t^2 + \exp(B_t)$, calculate $E[\,[Y,Y]_t\,]$, and simulate $[Y,Y]_t$ to find indications that the calculations are correct. (2 points)

Let $\hat{Z}_h^{\xi}(t)$ denote the estimated $Z(t)$, of a known SDE, with steplength h using numerical method ξ. Consider the SDEs $\{dX_t = X_t dt + X_t dB_t,\ X_0 = 1\}$ and $\{dY_t = -(\sqrt{1 - Y_t^2} + Y_t/2)dt - \sqrt{1 - Y_t^2} dB_t,\ Y_0 = 1\}$ (which during the course you have solved analytically, i.e. $X(t)$ and $Y(t)$ are known explicit).

2 Simulate $\hat{X}_h^{\xi}(t)$ and $\hat{Y}_h^{\xi}(t)$ using both Milstein and Euler methods to find indications that;

a) $E[|\hat{Z}_h^{Euler}(t) - Z(t)|]$ is growing in h,

b) $E[|\hat{Z}_h^{Milstein}(t) - Z(t)|] \leq E[|\hat{Z}_h^{Euler}(t) - Z(t)|]$,

if all conditions for the numerical methods are satisfied. (4 points)

Let the process $\{X_s\}_{s \in [t,T]}$ be the strong solution of the stochastic differential equation

$$dX_s = a(s, X_s)\, ds + b(s, X_s)\, dB_s,\ X_t = x, \quad s \in [t, T]. \tag{2}$$

Let $f : [t,T]\times\mathbb{R} \to \mathbb{R}$ be a function that is one time continuously differentiable in the first argument (t) and twice continuously differentiable in the second variable (x). Then, according to the Itô formula the stochastic process $\{f(s,X_s)\}_{s\in[t,T]}$ can be represented as

$$f(s,X_s) = f(t,X_t)+\int_t^s \frac{\partial f}{\partial u}(u,X_u)\,du+\int_t^s \frac{\partial f}{\partial x}(u,X_u)\,dX_u+\frac{1}{2}\int_t^s \frac{\partial^2 f}{\partial x^2}(u,X_u)\,d[X,X]_u, \tag{3}$$

where $\{[X,X]_u\}_{u\in[t,T]}$ is the quadratic variation process of $\{X_u\}_{u\in[t,T]}$. Suppose that the functions b and f are such that the stochastic integral process

$$\left\{\int_0^s b(u,X_u)\frac{\partial f}{\partial x}(u,X_u)\,dB_u\right\}_{s\in[t,T]}$$

is a martingale, with respect to the natural filtration of the Brownian motion.

3 Define the differential operator $\mathcal{A}$ as $\mathcal{A} = a(s,x)\frac{\partial}{\partial x}+\frac{1}{2}b^2(s,x)\frac{\partial^2}{\partial x^2}$. By taking expectations in (3), show that if the function $f(t,x)$ satisfies the partial differential equation $\{\frac{\partial f}{\partial t}(t,x)+(\mathcal{A}f)(t,x) = 0; f(T,x) = F(x)\}$, then the solution $f(t,x)$ can be represented as

$$f(t,x) = \mathbb{E}\{F\big(X_T\big)\,|\,X_t = x\},$$

where X_T is the solution of the SDE (2) evaluated at time T. (4 points)

4 Use the result of 6.3 to solve the partial differential equation

$$\{\frac{\partial f}{\partial t}(t,x)+\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0; f(T,x) = x^2\},$$

where $\sigma\in\mathbb{R}$ is a constant. (3 points)