

Stochastic Calculus II, 2007

Hand In 2

1. A stochastic process $\{X(t)\}_{t \geq 0}$ is called *self-similar* with index $\kappa > 0$ (*Hurst parameter*) if:

For $n \in \mathbb{N}$, $\lambda > 0$, and $t_1, \dots, t_n \geq 0$ it holds that

$$(X(\lambda t_1), \dots, X(\lambda t_n)) \stackrel{d}{=} (\lambda^\kappa X(t_1), \dots, \lambda^\kappa X(t_n))$$

It is an easy exercise to show that Brownian motion $\{B(t)\}_{t \geq 0}$ is a self similar process.

i) What is the Hurst parameter in the case of Brownian motion?

ii) Let $\tau_x = \inf\{t \geq 0 : B(t) = x\}$. Show that $\mathbf{P}(\tau_x > 1) = \mathbf{P}(|x|\sqrt{\tau_1} > 1)$.

(Note that x may be negative).

2. Let $\{X_t\}$ be a martingale and $g(x)$ a convex function, where $\mathbf{E}[|g(X_t)|] < \infty$. Show that $\{g(X_t)\}$ is a submartingale.

3. Let $X_t = e^{-\theta^2 t/2} \cosh(\theta B_t)$, where $\theta \in \mathbb{R}$.

i) Show that $\{X_t\}_{t \geq 0}$ is a martingale.

(Hint: what can you say about $-B_t$?)

ii) Find $E[X_t]$ and $Var(X_t)$.

iii) Is $\{X_t\}_{t \geq 0}$ uniformly integrable? Is it square integrable?