Hand-in no 4 Stochastic Calculus Part II

Let $\{B_t\}_{t\geq 0}$ be a Brownian motion and let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity 1 that is independent of B.

1 Let $T > 0$ be a constant time and define

$$M_t = \begin{cases} 0 & \text{for } t \leq T \\ B_{t-T} & \text{for } t > T \end{cases} \qquad \text{and} \qquad H_t = \begin{cases} B_t & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases}.$$

Let the filtration be $\mathcal{F}_t = \mathcal{F}_t^M \vee \mathcal{F}_t^N = \sigma(M(s) : s \leq t) \vee \sigma(N(s) : s \leq t)$.

(a) Show that $X_t = M_t + N_t$ is a semimartingale. (2 p)

(b) Establish that N_{t-} is predictable. (1 p)

(c) Show that $\int_0^t N_{s-}\, dX_s$ is well-defined for $t \geq 0$. (2 p)

(e) Now consider the filtration $\mathcal{F}_s = \mathcal{F}_{s+T}^M$. Find the expected value and the variance of $\int_0^t B_s\, dM_s$. (2 p)

2 Let $\alpha > 0$ and $\sigma > 0$ be constants. Consider the SDE $dX_t = -\alpha X_{t-}\, dt + \sigma\, dN_t$ (i.e., a non-Gaussian Langevin equation/Ornstein-Uhlenbeck process).

(a) Find the solution to the equation. (3 p)

(b) Make computer simulations to estimate $E(X_t)$ and $Var(X_t)$ for $\alpha = \sigma = 1$ and $X_0 = 0$. (2 p)

Let $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ be semimartingales.

3 (a) Motivate that $[X, Y]_t = \sum_{s\leq t} \Delta X_t \Delta Y_t$ when one of the processes X and Y has finite variation. (1 p)

(b) Show that if one of the processes X and Y is continuous with finite variation then $[X, Y] = 0$. (2 p)