Hand-in no 5 Stochastic, Calculus II

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity $\lambda = 1$.
Let $M_t = N_t - t$ be the compensated Poisson process.

1 Calculate $Y_t = \int_0^t M_{s-} dM_s$.
What is the quadratic covariation $[Y, M]_t$? (2p)

2 Let $\Lambda_t = e^{-t} 2^{N_t}$. Calculate $\mathbb{E}[\Lambda_t]$ and $Var(\Lambda_t)$.
Show that Λ_t is a martingale.
Simulate a trajectory of Λ_t. (2p)

3 As you know the Compound Poisson Process (CPP) $\{X_t\}_{t\geq 0}$ can be defined as

$$X_t = X_0 + \sum_{i=1}^{\infty} \mathbb{I}(\tau_i \leq t)\xi_i = X_0 + \sum_{i=1}^{N_t} \xi_i$$

where the ξ_i are iid random variables, independent of $\{N_t\}_{t\geq 0}$ (which has jump times τ_i).
Let $X_0 = 0$ and ξ_i be iid $N(0, 1)$.

a) Show, from $E[X_t|\mathcal{F}_s] = X_s$, that X_t is a martingale. (2p)

b) Calculate $E[X_t]$ and $Var(X_t)$. (2p)

c) Find estimates of $E[X_t]$ and $Var(X_t)$ by doing simulations. (2p)

d) Show that the characteristic function of X_t is given by

$$\phi(u) := E[e^{iuX_t}] = e^{t(\phi_\xi(u)-1)} = e^{t(e^{-u^2/2}-1)}.$$

The fact that the characteristic function of the ξ_i's is given by $\phi_\xi(u) = e^{-u^2/2}$ does not have to be shown. (3p)

e) Find the moment generating function of X_t. (1p)

f) Calculate $E[X_t^4]/3t$. (2p)

4 The Weibull distribution is widely used for modelling life times of objects. Its density and cdf, for $\alpha, \beta > 0$ and $\nu \in \mathbb{R}$, are given by

$$f(x;\alpha,\beta,\nu) = \begin{cases} 0 & x \leq \nu \\ \dfrac{\beta}{\alpha}\Big(\dfrac{x-\nu}{\alpha}\Big)^{\beta-1} \exp\Big\{-\Big(\dfrac{x-\nu}{\alpha}\Big)^{\beta}\Big\} & x > \nu \end{cases}$$

and

$$F(x;\alpha,\beta,\nu) = \begin{cases} 0 & x \leq \nu \\ 1 - \exp\Big\{-\Big(\dfrac{x-\nu}{\alpha}\Big)^{\beta}\Big\} & x > \nu \end{cases}$$

respectively (where β is the shape parameter, α is the scale parameter, and ν is the location parameter).

Just as in the CPP case let $\{X_t\}_{t \geq 0}$ be given by

$$X(t) = X(0) + \sum_{n=0}^{\infty} \mathbb{I}(\tau_i \leq t)\, \xi_i,$$

However, now let the interarrival times $\tau_{i+1} - \tau_i$ be iid Weibull(α, β, ν) = Weibull$(\alpha, 1, 0)$-distributed, for some $\alpha > 0$, and let the jumps ξ_i be iid $\mathrm{N}(\mu, \sigma)$-distributed.

a) Find the compensator $A(t)$ of $X(t)$. (2p)

b) Simulate one trajectory of $X(t)$ and one of $X(t) - A(t)$.
Also simulate estimates of $E[X(t)]$ and $E[M(t)] = E[X(t) - A(t)]$. (2p)

c) Give an example of a possible situation where one could use $X(t)$ as a modelling tool. (1p)