Hand-in no 6 Stochastic Calculus Part II

1 Let $W(t) = \mu t + B(t)$ for $t \geq 0$, where $\mu \in \mathbb{R}$ is a constant and B is a Brownian motion. Complete the details of Example 10.4 in Klebaner's book to find the probability distributions of

$$\left(\sup_{t\in[0,T]} W(t) \,\middle|\, W(T) = x \right) \quad \text{and} \quad \sup_{t\in[0,T]} W(t) \quad \text{for } T > 0. \quad \text{(2 points)}$$

2 As you know the compensator A of a Poisson process N with intensity $\lambda > 0$ is given by $A(t) = \lambda\, t$, making $M(t) = N(t) - \lambda\, t$ a martingale and thus on the average having zero deviation from zero.

Simulate a suitable long sample path of a Poisson process N with intensity $\lambda = 1$. Then discretize suitably $\{N(t_i)\}_{i=1}^{n}$ and peform a least-squares type of linear regression forced to pass through zero on the discretized data to estimate λ (pretending know that λ in unknown).

It is not necessarily the case that the above statistical procedure is very good, so don't be too worried if the result is not as sharp as you perhaps expected. However, I want you to prove that your least-squares estimator is unbiased!

As the data have unequal variances it can be expected that a weighted least-squares estimate should perform better than the naive unweighted one: Investigate what equations a search for optimal weights leads to! (3 points)

3 The CKLS equation is given by

$$dX(t) = (\alpha - \beta\, X(t))\, dt + \sigma\, X(t)^{\gamma}\, dB(t), \quad X(0) = x_0,$$

where $\beta \in \mathbb{R}$ and $\alpha, \sigma, \gamma, x_0 > 0$ are parameters. Find the stationary distribution π of X when $\gamma = 1/2$.

As you know, given $\sigma, \gamma > 0$, one might employ likelihood ratio techniques to estimate the parameters α and β of the drift. However, this requires that σ and γ are first estimated by another technique, namely by observing the quadratic variation process and estimating the values of σ and γ that makes this observed quadratic variation as close to the theoretical one $\int_0^t \sigma^2 X(s)^{2\gamma}\, ds$ (in least squares sense) as possible.

Simulate a suitable long and suitable densely discretized sample path of the CKLS process with parameter values $\alpha = \beta = \sigma = 1$, $\gamma = 1/2$ and $x_0 = \mathbf{E}\{\pi\}$. (If you coldn't find π above you may use $x_0 = 1$.) Plot the result.

Use the above indicated statistical procedure to estimate first σ and γ and then α and β, now pretending that the parameters are unknown. Again it is not necessarliy the case that the statistical procedure is very good, so don't be too worried if the result is not as good as you perhaps expected. (3 points)

4 Try to simulate the CKLS equation for $\gamma = 5$. What happens? What can one do about it? (2 points)