Lecture 7 December 11.15-12.00 AM

In these lecture notes we summarize some properties of compensators and sharp bracket processes. We will see important applications of these concepts next week in statistics for stochastic processes.

1. Compensator

Definition. Let $\{X(t)\}_{t\geq 0}$ be an adapted stochastic process with locally integrable (and in particular locally finite) variation. The *compensator* of X is the unique predictable (and in particular adapted) locally integrable process $\{A(t)\}_{t\geq 0}$ with $A(0) = 0$ such that $X(t) - A(t)$ is a local martingale.

Existence. As X has locally finite variation and is adapted and locally integrable, X is locally the difference between two increasing integrable adapted processes. However, an adapted increasing integrable process is a sub-martingale, and so X is the difference between two local sub-martingales. By Doob-Meyer decomposition (Theorem 8.4) a sub-martingale is the sum of a local martingale and an null at zero increasing predictable locally integrable process, and as the difference between two local martingales is a local martingale, it follows that X is the sum of a local martingale and the difference between two null at zero increasing predictable locally integrable processes, and thus the sum of a local martingale and a null at zero predictable locally integrable process.

2. Sharp bracket process

Definition. Let $\{X(t)\}_{t\geq 0}$ be a semi-martingale with locally integrable quadratic variation. The *sharp bracket process* or *predictable quadratic variation* $\{\langle X, X\rangle(t)\}_{t\geq 0}$ of X is the compensator of the quadratic variation process $[X, X]$.

Existence. As the quadratic variation process $[X, X]$ is increasing it is finite variation with variation process being equal to itself. As $[X, X]$ is locally integrable it follows that $[X, X]$ has locally integrable variation, and so the compensator of X exists.

Property 1. For X a locally square integrable martingale, the sharp bracket process $\langle X, X\rangle$ is the unique predictable locally integrable processes that makes $X^2 - \langle X, X\rangle$ a local martingale.

Proof of Property 1. As X is locally square integrable it follows from Doob's inequality [Equation (7.38)] and the Davis-Burkholder-Gundy inequality (Theorem 7.34) that X has locally integrable quadratic variation. Hence the sharp bracket process of X exists and is unique (as it is a compensator which are always unique). Further, as $\langle X, X\rangle$ is the compensator of $[X, X]$, we have that $[X, X] - \langle X, X\rangle$ is a local martingale. However, by integration by parts [Theorem 8.6 (which has as an assumption that X is locally square integrable, which Klebaner has forgotten to write out)], $X(t)^2 - [X, X](t) = X(0)^2 + 2\int_0^t X(s^-)\,dX(s)$, where the right-hand side is a local martingale as X is a local martingale (see top of page 216). As thus $[X, X] - \langle X, X\rangle$ and $X^2 - [X, X]$ are local martingales, so is their sum $X^2 - \langle X, X\rangle$.

Property 2. For a continuous local martingale the sharp bracket process exists and coincide with the quadratic variation process.

Proof of Property 2. As X is continuous it is locally bounded (as continuous functions over bounded intevals are bounded) and thus locally square integrable. Hence the sharp bracket process exists. As X is continuous so is $[X, X]$ [Equation (8.18)]. In particular $[X, X]$ is left-continuous, and thus predictable (as it is adaped). As X is locally square integrable $[X, X]$ is locally integrable (see above). As $[X, X]$ thus is predictable and locally integrable, with $[X, X] - [X, X] = 0$ being a martingale, it follows from Property 1 that $\langle X, X\rangle = [X, X]$.

Theorem 8.27. A martingale $\{M(t)\}_{t\in[0,T]}$ with $M(0) = 0$ (which Klebaner has forgotten to require in his statement of the result) is square integrable if and only if $\mathbf{E}\{[M, M](T)\} < \infty$ if and only if $\mathbf{E}\{\langle M, M\rangle(T)\} < \infty$. In any case we have $\mathbf{E}\{M(T)^2\} = \mathbf{E}\{[M, M](T)\} = \mathbf{E}\{\langle M, M\rangle(T)\}$.

Proof. We know that $\{M(t)\}_{t\in[0,T]}$ is square integrable if and only if $\mathbf{E}\{[M, M](T)\} < \infty$ by Doob's inequality and the Davis-Burkholder-Gundy inequality (see above). Letting $\{\tau_n\}_{n=1}^{\infty}$ be a localizing sequence of stopping times making $[M, M](t\wedge\tau_n) - \langle M, M\rangle(t\wedge\tau_n)$ a martingale we may take expected values [remebering that $\langle M, M\rangle(0) = 0$ as $\langle M, M\rangle$ is a compensator] to obtain $0 = \mathbf{E}\{[M, M](T\wedge\tau_n) - \langle M, M\rangle(T\wedge\tau_n)\}$. Now, sending $n \to \infty$ so that $\tau_n \to \infty$, the facts that $[M, M]$ and $\langle M, M\rangle$ are increasing (see below) together with monotone convergence gives $\mathbf{E}\{[M, M](T\wedge\tau_n)\} \to \mathbf{E}\{[M, M](T)\}$ and $\mathbf{E}\{\langle M, M\rangle(T\wedge\tau_n)\} \to \mathbf{E}\{\langle M, M\rangle(T)\}$. Hence, if one of $\mathbf{E}\{[M, M](T)\}$ and $\mathbf{E}\{\langle M, M\rangle$

$(T)\}$ are finite, we have $0 = \mathbf{E}\{[M,M](T\wedge\tau_n) - \langle M,M\rangle(T\wedge\tau_n)\} \to \mathbf{E}\{[M,M](T)\} - \mathbf{E}\{\langle M,M\rangle(T)\}$, so that both of them must be finite.

In any case we see that $\mathbf{E}\{[M,M](T)\} = \mathbf{E}\{\langle M,M\rangle(T)\}$, while $\mathbf{E}\{[M,M](T)\} = \mathbf{E}\{M(T)^2\}$ follows from the fact that $M^2 - [M,M]$ is a martingale when M is square integrable (Theorem 7.27).

3. Sharp cobracket process

Definition. The *sharp cobracket process* $\{\langle X,Y\rangle(t)\}_{t\geq 0}$ between two semi-martingales X and Y with locally integrable quadratic variation is the compensator of the quadratic covariation process $[X,Y]$.

Existence. The sharp cobracket process between two semi-martingales X and Y with locally integrable quadratic variation exists by polarization $[X,Y] = \frac{1}{4}([X+Y,X+Y] - [X-Y,X-Y])$ where $X+Y$ and $X-Y$ are semi-martingales whose quadratic variations are locally integrable, because X and Y have locally and $\sum_{i=1}^n (X(t_i) \pm Y(t_i) - X(t_{i-1}) - (\pm)Y(t_{i-1}))^2 \leq 2\sum_{i=1}^n (X(t_i) - X(t_{i-1}))^2 + 2\sum_{i=1}^n (Y(t_i) - Y(t_{i-1}))^2$.

Property 1. $\langle X,X\rangle$ is increasing (because in the compensator of an increasing process, which is increasing by the Doob-Meyer decomposition).

Property 2 (Polarization). $\langle X,Y\rangle = \frac{1}{4}(\langle X+Y,X+Y\rangle - \langle X-Y,X-Y\rangle)$ [because as $[X+Y,X+Y] - \langle X+Y,X+Y\rangle$ and $[X-Y,X-Y] - \langle X-Y,X-Y\rangle$ are local martingales, so is $[X,Y] - \frac{1}{4}(\langle X+Y,X+Y\rangle - \langle X-Y,X-Y\rangle) = \frac{1}{4}([X+Y,X+Y] - [X-Y,X-Y]) - \langle X-Y,X-Y\rangle)$. As $\frac{1}{4}(\langle X+Y,X+Y\rangle - \langle X-Y,X-Y\rangle)$ is predictable and locally integrable (as $\langle X+Y,X+Y\rangle$ and $\langle X-Y,X-Y\rangle$ are, being compensators), it follows that $\frac{1}{4}(\langle X+Y,X+Y\rangle - \langle X-Y,X-Y\rangle)$ must be the unique compensator of $[X,Y]$, and thus is the sharp cobracket process $\langle X,Y\rangle$].

Property 3 (Symmetry). $\langle X,Y\rangle = \langle Y,X\rangle$ (by polarization).

Property 4 (Bilinearity). $\langle \alpha_1X_1+\alpha_2X_2, \beta_1Y_1+\beta_2Y_2\rangle = \alpha_1\beta_1\langle X_1,Y_1\rangle + \alpha_2\beta_1\langle X_2,Y_1\rangle + \alpha_1\beta_2\langle X_1,Y_2\rangle + \alpha_2\beta_2\langle X_2,Y_2\rangle$ (as $[\alpha_1X_1+\alpha_2X_2, \beta_1Y_1+\beta_2Y_2] = \alpha_1\beta_1[X_1,Y_1] + \alpha_2\beta_1[X_2, Y_1] + \alpha_1\beta_2[X_1,Y_2] + \alpha_2\beta_2[X_2,Y_2]$, so that the sum of the compensators of the righ-hand side is the compensator of the left-hand side, as it is predictable and locally integrable and makes the left-hand side a local martingale).

Property 5. $\langle X, Y\rangle = 0$ if X and Y are finite variation processes and one of them is continuous [as writing $X^{\rm c}$ and $X^{\rm cm}$ for the continuous part and continuous local martingale part of a semi-martingale, respectively, we have $[X,X]^c(t) = \langle X^{\rm cm}, X^{\rm cm}\rangle(t)$ (Equation 8.49), which by polarization gives $[X,Y]^c(t) = \langle X^{\rm cm}, Y^{\rm cm}\rangle(t)$, and here one of $X^{\rm cm}$ and $Y^{\rm cm}$ is zero as either $X = X^{\rm cm}$ or $Y = Y^{\rm cm}$ and a finite variation semi-martingale has zero continuous local martingale part (Corollry 8.30)].

Property 6. For M a locally square integrable martingale and H a predictable process such that $\int_0^t H(s)^2\, d\langle M, M\rangle(s) < \infty$ for all t the stochastic integral $\int_0^t H(s)\, dM(s)$ exists and is a local martingale.

Property 7. $\langle \int_0^t H(s)\, dX(s), \int_0^t K(s)\, dY(s)\rangle = \int_0^t H(s)K(s)\, d\langle X, Y\rangle(s)$ for locally square integrable martingales X and Y such that $\int_0^t H(s)^2\, d\langle X, X\rangle(s) < \infty$ and $\int_0^t Y(s)^2$ $d\langle Y, Y\rangle(s) < \infty$.

Property 8 (Isometry). For M a locally square integrable martingale and H a predictable process such that $\mathbf{E}\big\{ \int_0^t H(s)^2\, d\langle M, M\rangle(s)\big\} < \infty$ for all t we have that $\int_0^t H(s)\, dM(s)$ is a square integrable martingale with $\langle \int_0^t H(s)\, dM(s), \int_0^t H(s)\, dM(s)\rangle = \int_0^t H(s)H(s)\, d\langle M, M\rangle(s)$ and $\mathbf{E}\big\{(\int_0^t H(s)\, dM(s)^2\big\} = \mathbf{E}\big\{ \int_0^t H(s)^2\, d\langle M, M\rangle(s)\big\}$ (by using Theorem 8.27).

4. Examples

Poisson process. For the Poisson process N we have $\langle N, N\rangle(t) = t$ as this process is predictable [being continuous and adapted (as it is non-randon)] and $[N, N] = N$ with $N(t) - t$ being a martingale, so that $[N, N](t) - t$ is also a martingale.

Brownian motion. For the Brownian motion B we have $\langle B, B\rangle(t) = [B, B](t) = t$ as B is a continuous locally square integrable martingale.