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Suggested solution for exam in
MSA830: Statistical Analysis and Experimental Design
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1. (a) Assuming that the outcome of each game is independent, the probability can be computed using the Binomial distribution:

$$\binom{10}{8} 0.4^8(1-0.4)^{10-8} = \frac{10 \cdot 9}{1 \cdot 2} 0.4^8 0.6^2 = 0.01062$$

(b) The probability that she wins 9 games is

$$\binom{10}{9} 0.4^1(1-0.4)^{10-9} = \frac{10}{1} 0.4^9 0.6^1 = 0.00157$$

and the probability that she wins 10 games is

$$\binom{10}{10} 0.4^{10}(1-0.4)^{10-10} = 0.4^{10} = 0.00010$$

so the total probability that she wins 8 or more games is

$$0.01062 + 0.00157 + 0.00010 = 0.01229$$

(c) The first of these four games can be won by anybody but Lisa, and the probability for this happening is 0.6. In the second game, any of the three friends who have not yet won can win, and the probability that any of them wins is $3 \cdot 0.15 = 0.45$. Similarly, the probability that any of the two remaining non-winning friends win in the third game is 0.3, while in the fourth game the last non-winning friend must win, and that has a probability of 0.15. In conclusion, the probability becomes

$$0.6 \cdot 0.45 \cdot 0.3 \cdot 0.15 = 0.01215$$

An alternative way of reasoning is the following: The probability is 0.15^4 for observing a specific sequence of winners, none of whom is Lisa. As there are 4! such sequences, the probability in question becomes

$$4! \cdot 0.15^4 = 0.01215$$

2. (a) As the standard deviations of the two normal distributions may be different, the expected change in strength has (approximate) distribution

$$\text{t}\left(3.35 - 3.21, \nu, \log\sqrt{\frac{0.11^2}{14} + \frac{0.14^2}{9}}\right) = \text{t}\left(0.14, \nu, \log(0.05515)\right)$$

where the degrees of freedom is computed as

$$\nu = \frac{\left(\frac{0.11^2}{14} + \frac{0.14^2}{9}\right)^2}{\frac{(0.11^2/14)^2}{14-1} + \frac{(0.14^2/9)^2}{9-1}} = 14.23$$

Using the tables, we see that a 95% credibility interval for the standard t-distribution with 14 degrees of freedom is $[-2.1448, 2.1448]$, so a 95% credibility interval for our the expected change in strength is

$$[0.14 - 2.1448 \cdot 0.05515, 0.14 + 2.1448 \cdot 0.05515] \approx [0.02, 0.26]$$

(b) The logged scale for the measured strength of material B has distribution

$$\text{ExpGamma}\left(\frac{12-1}{2}, \frac{12-1}{2}0.08^2, -2\right) = \text{ExpGamma}\,(11/2, 0.0704/2, -2)$$

A 95% credibility interval for the standard deviation thus becomes

$$\left[\sqrt{\frac{0.0704}{\chi^2_{0.025,11}}}, \sqrt{\frac{0.0704}{\chi^2_{0.975,11}}}\right] = \left[\sqrt{\frac{0.0704}{21.92}}, \sqrt{\frac{0.0704}{3.816}}\right] = [0.057, 0.136]$$

(c) If all the standard deviations of the normal distributions are assumed equal, we can use the theory for linear models. The logged scale in a linear model has distribution

$$\text{ExpGamma}\left(\frac{n-k}{2}, \frac{1}{2}SS, -2\right)$$

where n is the total number of observations, so in our case

$$n = 14 + 12 + 9 + 15 + 10 = 60,$$

k is the number of beta parameters in the model, in our case 5 (as we have 5 groups of observations), and where SS is the sum of squares of residuals, which in our case can be computed as

$$SS = (14-1)\cdot 0.11^2 + (12-1)\cdot 0.08^2 + (9-1)\cdot 0.14^2 + (15-1)\cdot 0.15^2 + (10-1)\cdot 0.09^2 = 0.7724$$

So the logged scale has distribution

$$\text{ExpGamma}\left(\frac{55}{2}, \frac{0.7724}{2}, -2\right)$$

Unfortunately, our table does not contain a line for 55 degrees of freedom, but making a rough interpolation using the available values, we get approximately $\chi^2_{0.975,55} \approx 77$ and $\chi^2_{0.025,55} \approx 36$. With this, 95% credibility interval for the standard deviation becomes

$$\left[\sqrt{\frac{0.7724}{\chi^2_{0.025,55}}}, \sqrt{\frac{0.7724}{\chi^2_{0.975,55}}}\right] \approx \left[\sqrt{\frac{0.7724}{77}}, \sqrt{\frac{0.7724}{36}}\right] \approx [0.10, 0.15]$$

3. We use the extra information that 0.7% of all baches have high concentration of the bacteria, and 0.7% of all baches have medium concentration, while the rest, i.e., 98.6% of the baches, have low concentration. we get, usign Bayes Theorem:

$$\begin{aligned}
& \pi(\text{High c.}|\text{Chem. pres.}) \\
= & \frac{\pi(\text{Chem. pres.}|\text{High c.})\pi(\text{High c.})}{\pi(\text{Chem. pres.})} \\
= & \frac{\pi(\text{Chem. pres.}|\text{High c.})\pi(\text{High c.})}{\pi(\text{C. pres.}|\text{H. c.})\pi(\text{H. c.}) + \pi(\text{C. pres.}|\text{M. c.})\pi(\text{M. c.}) + \pi(\text{C. pres.}|\text{L. c.})\pi(\text{L. c.})} \\
= & \frac{0.9 \cdot 0.007}{0.9 \cdot 0.007 + 0.35 \cdot 0.007 + 0.05 \cdot 0.986} \\
= & 0.109
\end{aligned}$$

So the probability is about 11%. As there was some information missing in the text of this question, special consideration has been used when grading it.

4. (a) The data is clearly paired, so that we should analyze the differences

$$25, 10, 18, -5, 33, 15, 11$$

These numbers have mean 15.28571 and variance 145.5714. Under the assumptions mentioned below, the expected difference in customer satisfaction has the distribution

$$\text{t}\left(15.28571, 7-1, \log\left(\sqrt{145.5714/7}\right)\right) = \text{t}\left(15.28571, 6, \log(4.560254)\right)$$

and a 90% credibility interval becomes

$$[15.28571 - 1.9432 \cdot 4.560254, 15.28571 + 1.9432 \cdot 4.560254+] \approx [6.4, 24.1]$$

(b) We assume above that the differences are independent observations and that they come from a normal distribution.

(c) One can use the non-parametric Wilcoxon signed-rank test. To compute the test statistic, we first rank, or order, the observations according to their absolute values:

$$-5, 10, 11, 15, 18, 25, 33$$

We then sum the ranks of the negative observations to get W_- and the ranks of the positive observations to get W_+, so that $W_- = 1$ and

$$W_+ = 2 + 3 + 4 + 5 + 6 + 7 = 27$$

The test statistic is the smallest of these two numbers, i.e., 1.

5. (a) A fractional factorial experimental plan is

A	B	C	D	E	F
-	-	-	-	-	-
-	-	-	-	+	+
-	-	-	+	-	+
-	-	-	+	+	-
-	-	+	-	-	+
-	-	+	-	+	-
-	-	+	+	-	-
-	-	+	+	+	+
-	+	-	-	-	+
-	+	-	-	+	-
-	+	-	+	-	-
-	+	-	+	+	+
-	+	+	-	-	-
-	+	+	-	+	+
-	+	+	+	-	+
-	+	+	+	+	-
+	-	-	-	-	+
+	-	-	-	+	-
+	-	-	+	-	-
+	-	-	+	+	+
+	-	+	-	-	-
+	-	+	-	+	+
+	-	+	+	-	+
+	-	+	+	+	-
+	+	-	-	-	-
+	+	-	-	+	+
+	+	-	+	-	+
+	+	-	+	+	-
+	+	+	-	-	+
+	+	+	-	+	-
+	+	+	+	-	-
+	+	+	+	+	+

The design is obtained as a full factorial design for A, B, C, D, E, while the column for F has been obtained as the product of all the previous columns. This design has the property that the inference from the experiment will not be influenced by which factor is assigned to which column in the plan.

(b) The two most important things are probably:

- George should make sure to randomize the order in which he does the experiments, so that factors he cannot control are less likely to have a systematic effects on his results.
- For factors he can control (but not among the factors A, B, C, D, E, F) he should in general try to keep them as constant as possible during all the 32 experiments. (He might also consider blocking for some of these factors).

6. (a) We get

$$\begin{aligned}
SS_{\text{DesignType}} &= 20\Big((8.3-8.43.75)^2+(11.15-8.4275)^2\\
&\quad+(7.5-8.4375)^2+(6.8-8.4375)^2\Big)=218.7375\\
SS_{\text{PaperThickness}} &= 40\Big((7.425-8.4375)^2+(9.45-8.4375)^2\Big)=82.0125\\
SS_{\text{PlaneConstructor}} &= 40\Big((8.325-8.4375)^2+(8.55-8.4375)^2\Big)=1.0125\\
SS_{\text{Total}} &= 79\cdot 17.4644=1379.688
\end{aligned}$$

(b) The table becomes

	SS	D.f.	M.sq.	F	p
DesignType	218.7375	3	72.9125	5.005	$p < 0.01$
PapterThickness	82.0125	1	82.0125	5.630	$0.01 < p < 0.025$
PlaneConstructur	1.0125	1	1.0125	0.0695	$p > 0.25$
Residuals	1077.926	74	14.56657		
Total	1379.688	79			

We get that the design type has a significant effect on the flight distance; from the averages we can see that Type2 appears to give the longest flight distance. The paper thickness also has a significant effect; from the averages we see that the Thin paper gives the longest expected flight distance. However, from the given data, we can not conclude that whether Eric or Axel constructed the planes has an influence on the flight distance.

(c) We first compute the sum of squares including all the three factors above and interaction, let us for short call it SS_{All}. It can be computed from the averages given in the table and the grand average:

$$\begin{aligned}
SS_{\text{All}} &= 5\Big((8.2-8.4375)^2+(8.6-8.4375)^2+(8.2-8.4375)^2+(2.6-8.4375)^2\\
&\quad+(7.6-8.4375)^2+(12.2-8.4375)^2+(5.2-8.4375)^2+(6.8-8.4375)^2\\
&\quad+(10.2-8.4375)^2+(10.4-8.4375)^2+(10.0-8.4375)^2+(8.4-8.4375)^2\\
&\quad+(7.2-8.4375)^2+(13.4-8.4375)^2+(6.6-8.4375)^2+(9.4-8.4375)^2\Big)\\
&= 510.4875
\end{aligned}$$

For the $SS_{\text{Residuals}}$ in the new table, we now get

$$SS_{\text{Residuals}} = SS_{\text{Total}} - SS_{\text{All}} = 1379.688 - 510.4875 = 869.2005$$

As the old value for residuals should be split into the new value for residuals and $SS_{\text{Interaction}}$, we get

$$SS_{\text{Interaction}} = 1077.926 - 869.2005 = 208.7255$$

We now get the following ANOVA table:

	SS	D.f.	M.sq.	F	p
DesignType	218.7375	3	72.9125		
PapterThickness	82.0125	1	82.0125		
PlaneConstructur	1.0125	1	1.0125		
Interaction	208.7255	10	20.8725	1.5368	$0.1 < p < 0.25$
Residuals	869.2005	64	13.5812		
Total	1379.688	79			

According to the p-value, we should not include interaction in the analysis.

(d) The sums of squares $SS_{\text{DesignType}}$, $SS_{\text{PaperThickness}}$, and $SS_{\text{PlaneConstructor}}$ will be computed exactly as in part (a), except for the factor 5, so we get

$$\begin{aligned} SS_{\text{DesignType}} &= 218.7375/5 = 43.7475 \\ SS_{\text{PaperThickness}} &= 82.0125/5 = 16.4025 \\ SS_{\text{PlaneConstructor}} &= 1.0125/5 = 0.2025 \end{aligned}$$

The new SS_{Total} will in fact correspond to the SS_{All} computed above, again except for a factor 5, so we get

$$SS_{\text{Total}} = 510.4875/5 = 102.0925$$

Finding the new $SS_{\text{Residuals}}$ by subtraction (or by dividing the old value for $SS_{\text{Interaction}}$ by 5) this results in the table

	SS	D.f.	M.sq.	F	p
DesignType	43.7475	3	14.5825	3.49	$0.05 < p < 0.1$
PapterThickness	16.4025	1	16.4025	3.93	$0.05 < p < 0.1$
PlaneConstructur	0.2025	1	0.2025	0.04	$p > 0.25$
Residuals	41.745	10	4.1745		
Total	102.0975	15			

We see that, with this analysis, none of the factors seem to have a significant effect on the flight distance.

7. (a) It is dataB: The normal probability plot indicates that the data is divided into two separate groups, and this shown in the histogram for dataB.

   (b) The figure does NOT show that this is an unsuiltable model: Indeed it just shows that the data seems to fall into two groups, which is what Ingela plans to use in here analysis. If figure 1 had been a plot of the *residuals* in the analysis, there would have been strong reasons to believe that the analysis was unsuitable.