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Suggested solution for re-exam in
MSA830: Statistical Analysis and Experimental Design
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NOTE: The notation of this document has been updated February 2012.

1. (a) The probability that none of his experiments were successful is

$$(1-0.7)(1-0.4)(1-0.4) = 0.3 \cdot 0.6 \cdot 0.6 = 0.108$$

Thus, the probability that at least one was successful is $1 - 0.108 = 0.892$.

(b) Lets use the notation

success	The experiment was a success
not sucess	The experiment was not a success
type X	The experiment was of type X
other type	The exeriment was of another type (i.e., Y or Z)

We can then use Bayes formula to write

$$\begin{aligned}
&\pi(\text{type X} \mid \text{success}) \\
=\ & \frac{\pi(\text{success} \mid \text{type X})\pi(\text{type X})}{\pi(\text{success})} \\
=\ & \frac{\pi(\text{success} \mid \text{type X})\pi(\text{type X})}{\pi(\text{success} \mid \text{type X})\pi(\text{type X}) + \pi(\text{success} \mid \text{other type})\pi(\text{other type})} \\
=\ & \frac{0.7 \cdot \frac{1}{3}}{0.7 \cdot \frac{1}{3} + 0.4 \cdot \frac{2}{3}} = 0.466
\end{aligned}$$

So the probability is approximately 47%.

2. If we assume that the rate of tornatoes is stable from year to year, and that each tornado appears independently from other tornadoes, we can assume that the number of tornadoes during 2010 is Poisson distributed with a rate of 5.6.

(a) The probability of observing x for a Poisson distribution with rate 5.6 is

$$\pi(x) = \exp(-5.6)\frac{1}{x!}5.6^x$$

Thus the probability of observing 0 or 1 is

$$\begin{aligned}
\pi(0) + \pi(1) &= \exp(-5.6)\frac{1}{0!}5.6^0 + \exp(-5.6)\frac{1}{1!}5.6^1 = \exp(-5.6)(1 + 5.6) \\
&= 0.003697864 \cdot 6.6 = 0.0244
\end{aligned}$$

The probability is about 2.4%.

(b) We can here use an approximation by the normal distribution. The probability of observing an x of 12 or more is approximately equal to the probability that a variable with a standard normal distribution is above

$$\frac{11.5 - 5.6}{\sqrt{5.6}} = 2.493$$

According to the table for the standard normal distribution, this probability is $1 - 0.99361 = 0.00639$, or about 0.6%.

(c) Under the assumption, the probability of observing 12 tornadoes "or something more extreme" compared to what would be expected under the assumption is twice the probability computed in (b), i.e., about 1.2%. As this is less than 5%, it is customary to *reject* the the the assumptions are correct. Thus one would say that tornadoes do not happen independently, or, possibly, that there is a trend in the weather or climate that is changing the rate of tornadoes.

3. (a) According to the description of the problem, it is natural to use as data the amount cut for seed A minus the amount cut for seed B at each location. The observed data then becomes 0, 4, 1, 0, and 7. The average $\overline{x}$ and sample variance s^2 of these values are

$$\overline{x} = 2.4$$

and

$$s^2 = 9.3$$

A reasonable assumption is that the differences come from a normal distribution with expectation μ and logged scale λ. We then get that

$$\mu \mid \text{data} \sim \text{t}(\overline{x}, 5 - 1, \log \sqrt{\frac{s^2}{5}}) = \text{t}(2.4, 4, \log \sqrt{1.86})$$

So the expected difference is 2.4. As a 95% credibility interval for the standard t distribution with 4 degrees of freedom is $[-2.7764, 2.7764]$, a 95% credibility interval for μ is

$$[2.4 - 2.7764 \cdot \sqrt{1.86}, 2.4 + 2.7764 \cdot \sqrt{1.86}] = [-1.39, 6.19]$$

(b) The assumtions are that that the differences come from a normal distribution, and that each of these differences are independent obserations from this normal distribution.

(c) As the length of the credibility interval contains the factor $\frac{1}{\sqrt{n}}$ where n is the number of observations, we can roughly estimate that when n increases by a factor of 4, the length of the credibility interval is halved. As the old length was $8.19 - (-1.39) = 7.58$, the rough guess is 3.79. (More precisely, the length of the credibility interval would be $2 \cdot t_{0.025,20} \cdot \sqrt{s^2_{NEW}/20}$ where $t_{0.025,20} = 2.086$, and where s^2_{NEW} would be the sample variance of the new data. As our best guess for this value is the sample variance of the old data, our best guess for the length of the new credibility interval would be $2 \cdot 2.086 \cdot \sqrt{9.320} = 2.844925$.)

4. Emmett should use the design matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.3 & 0.09 \\ 1 & 0.6 & 0.36 \\ 1 & 1 & 1 \\ 1 & 1.5 & 2.25 \\ 1 & 2 & 4 \end{bmatrix}$$

5. (a) We need to assume that the measurements are normally distributed with expectation μ and logged scale λ. Then the logged scale λ of the observations has distribution

$$\tau \mid \text{data} \sim \text{ExpGamma}\left(\frac{7-1}{2}, \frac{7-1}{2}0.0235, -2\right) = \text{Gamma}\left(\frac{6}{2}, \frac{0.141}{2}, -2\right)$$

As a 95% credibility interval for the $\chi^2(6)$ distribution is [1.237, 14.449], a 95% credibility interval for e^λ | data is

$$\left[\sqrt{0.141/14.449}, \sqrt{0.141/1.237}\right] = [0.09878, 0.3376]$$

Thus a 95% credibility interval for the precision $1/(e^\lambda)^2$ becomes

$$[1/0.3376^2, 1/0.09878^2] = [8.77, 102]$$

(b) The test statistic becomes

$$\frac{0.0921}{0.0235} = 3.919149$$

which should be compared with an F distribution with 10 and 6 degrees of freedom. From the tables for the F distribution, we see that the probability for a variable with this distribution to be above 3.91 is between 0.05 and 0.1, so the p-value for the test is in the interval between 0.1 and 0.2. As the p-value is above 0.05, we do not reject the null hypothesis that the two population variances are equal.

6. (a) If the four factors are names A, B, C, and D, respectively, and if their two levels are named "+" and "-", then a possible fractional factorial design for 8 experiments would be

A	B	C	D
-	-	-	-
-	-	+	+
-	+	-	+
-	+	+	-
+	-	-	+
+	-	+	-
+	+	-	-
+	+	+	+

Here, the last column, for D, has been produced by multiplying the three previous columns: $D = ABC$.

(b) As there are many other factors than the ones listed by Pernilla that will affect her sales, which she cannot controle, and some of which vary with time, it is important that she performs the 8 experiments in a randomized order. For all factors that Pernilla can control, she should strive to keep them as constant as possible over the 8 days of the trial period.

7. (a) For the different sums of squares we get

$$
\begin{aligned}
SS_{Nutrient} &= 8((7.0375 - 7.0292)^2 + (7.925 - 7.0292)^2 + (6.125 - 7.0292)^2) = 12.96083 \\
SS_{Temp} &= 12((6.175 - 7.0292)^2 + (7.8833 - 7.0292)^2) = 17.50973 \\
SS_{Ph} &= 12((6.725 - 7.0292)^2 + (7.3333 - 7.0292)^2) = 2.220173 \\
SS_{Total} &= 23 \cdot 2.557808 = 58.82958 \\
SS_{Residuals} &= 58.82958 - 12.96083 - 17.50973 - 2.220173 = 26.13885
\end{aligned}
$$

This leads to the ANOVA table

	SS	D.f.	M.sq.	F	p
Nutrient	12.96083	2	6.480415	4.710532	$0.01 < p < 0.025$
Temp	17.50973	1	17.50973	12.7276	$p < 0.01$
Ph	2.220173	1	2.220173	1.629358	$0.1 < p < 0.25$
Residuals	26.13885	19	1.375729		
Total	58.82958	23			

(b) We have to assume that the data can be modelled by a linear model without interaction: In other words, that observations can be described as the sum of effects based on the three predictors plus *independent*, *normally distributed* errors with *equal variances*.

(c) As the p-value for both Nutrient and Temperature are well below 0.05, one could conclude that these factors influence the growth rates significantly. However, the Ph value does not seem to have a significant influence. Looking at the averages, one can see that Nutrient B seems to give the best growth rate, and that the low temperature also seem to give the best growth rate.

(d) The new ANOVA table becomes:

	SS	D.f.	M.sq.	F	p
Nutrient	12.96083	2	6.480415	2.969359	$0.05 < p < 0.1$
Ph	2.220173	1	2.220173	1.017293	$0.25 < p$
Residuals	43.64858	20	2.182429		
Total	58.82958	23			

(e) The partial ANOVA table is:

	SS	D.f.	M.sq.	F	p
Nutrient	12.96083	2	6.480415		
Temp	17.50973	1	17.50973		
Ph	2.220173	1	2.220173		
Temp:Nutr		2			
Residuals		17			
Total	58.82958	23			