Problem 1
Let $(X_i, i = 1, \dots, n)$ be a normally distributed random sample with $X_i \sim \mathcal{N}(\theta, \sigma^2)$ for $i = 1, \dots, n$. Denote the sample mean by $\bar{X} := 1/n \sum_{i=1}^{n} X_i$ and assume that σ^2 is known.

(a) Show that $\bar{X}$ is a minimal sufficient statistic for θ.

(b) Show that the family of normal pdfs with σ^2 known is an exponential family.

(c) Show that $\bar{X}$ is a complete statistic for θ.

(d) Derive estimators for θ and σ^2 with the method of moments and show if the estimators are unbiased, where you are allowed to use your knowledge about S^2.

(e) Derive the likelihood function for θ. Compute the maximum likelihood estimator for θ.

(f) Define the mean squared error of an estimator of a parameter and compute it for $\bar{X}$.

(22 points)

Problem 2
Let $(X_i, i = 1, \dots, n)$ be a normally distributed random sample with $X_i \sim \mathcal{N}(\theta, \sigma^2)$ for $i = 1, \dots, n$. Assume that σ^2 is known and consider the hypothesis test $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

(a) Give the definition of a likelihood ratio statistic λ and use it to define a likelihood ratio test.

(b) Compute the likelihood ration test for the given scenario by giving the rejection region of the test with respect to $c \in [0, 1]$.

(c) Give the definition of a monotone likelihood ration and show that the family of normal distributions with σ^2 known has a monotone likelihood ratio.

(d) Define what a uniformly most powerful level α test is and construct such a test for the given scenario by using $\bar{X}$ as estimator for θ, where you should not forget to show that your constructed test has the desired properties.

(e) Construct a valid p-value for the estimator $\bar{X}$ of θ within the given test scenario.

(16 points)

Problem 3

Let $(X_i, i = 1, \dots, n)$ be a normally distributed random sample with $X_i \sim \mathcal{N}(\theta, \sigma^2)$ for $i = 1, \dots, n$. Denote the sample mean by $\bar{X} := 1/n \sum_{i=1}^n X_i$ and assume that σ^2 is known.

(a) Define what a pivot is and show that $\bar{X} - \theta$ is a pivot for θ.

(b) For a given $\alpha \in [0, 1]$, compute the shortest possible $1 - \alpha$ confidence interval for θ by using the pivot $\bar{X} - \theta$.

(9 points)

Problem 4

Prove the following statements:

(a) Define first what a best unbiased estimator W of $\tau(\theta)$ is. Show further that the best unbiased estimator W of $\mathbb{E}_\theta(W)$ satisfies that

$$\mathsf{Cov}_\theta(W, U) = 0$$

for all $\theta \in \Theta$, where Θ denotes the parameter space and U is any unbiased estimator of 0.

(b) For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a level α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. For each $x \in \mathcal{X}$, define the set $C(x)$ in the parameter space by

$$C(x) := \{\theta_0 \in \Theta, x \in A(\theta_0)\},$$

where $\mathcal{X}$ denotes the sample space. Show that the random set $C(X)$ is a $1 - \alpha$ confidence set.

(c) Give the definition of a consistent sequence of estimators of a parameter θ. Show that $(W_n, n \in \mathbb{N})$ is a consistent sequence of estimators of θ if for all $\theta \in \Theta$

(a) $\lim_{n\to\infty} \mathsf{Var}_\theta W_n = 0$ and

(b) $\lim_{n\to\infty} \mathsf{Bias}_\theta W_n = 0$.

(13 points)