CHALMERS | UNIVERSITY OF GOTHENBURG

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Statistical Inference Principles – Spring 2016

Assignment 1

1. Let X be a random variable with finite mean μ and variance σ^2. Furthermore, let $(X_i, i = 1, \dots, n)$ be a random sample with the same properties. Define the two random variables

$$\bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

(a) Show that $\bar{X}$ and S^2 are unbiased estimators for the mean and the variance of X.

(b) Let M_X denote the *moment generating function* (mgf) of X given by

$$M_X(t) := \mathbb{E}(\exp(tX)), \quad t \in \mathbb{R}.$$

Show that the mgf $M_{\bar{X}}$ of $\bar{X}$ is given by

$$M_{\bar{X}}(t) = \left(M_X(n^{-1}t)\right)^n.$$

2. Let $(X_i, i = 1, \dots, n)$ be a random sample from a Bernoulli(p) distribution, $p \in (0, 1)$.

(a) Show that the family of Bernoulli(p) pmfs for $p \in (0, 1)$ is an exponential family with $h(x) = \mathbb{1}_{\{0,1\}}(x)$, $c(p) = 1 - p$, $k = 1$, $w_1(p) = \log \frac{p}{1-p}$, and $t_1(x) = x$.

(b) Show first that $T_1(X_1, \dots, X_n) := \sum_{j=1}^{n} t_1(X_j)$ is binomial(n, p) distributed. Furthermore, show that the family of binomial(n, p) pmfs is an exponential family with the same parameters as in (a) except for $H(x) = \mathbb{1}_{\{0,\dots,n\}}(x)\binom{n}{x}$ and $C(p) = c(p)^n = (1 - p)^n$.

3. Suppose that the sequence of random variables $(X_i, i \in \mathbb{N})$ converges in probability to the random variable X and that $h : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous mapping. Show that $(h(X_i), i \in \mathbb{N})$ converges in probability to $h(X)$.

Deadline: Monday, January 25, 2016
Webpage: `http://www.math.chalmers.se/Stat/Grundutb/GU/MSF100/S16/`
Requirement: 50% of the exercises solved, two presentations in the exercise class