2018-03-17

MSF 100 & MVE 326
Statistical Inference Principles

Please make sure before you start:

- The exam is on *March 17, 2018, 8:30 – 12:30.*
- The examiner is *Annika Lang*, Mathematical Sciences, Chalmers, phone: 0317725356.
- Andreas Petersson (0317725325) visits the exam at 9:30 and 11:00.
- You are *allowed* to use during the exam *4 pages* (2 sheets, double-sided) of *hand-written* notes and a simple calculator.
- The maximum number of points that can be achieved is 60. You need *30 points* to pass the exam (for GU: 30 points for G and 45 points for VG, for Chalmers 30 points for 3, 40 points for 4, and 50 points for 5).
- Read all problems carefully before you start to work on the exam.
- Write your solutions in detail and readable. If you use theorems, definitions, etc., from the lecture, cite the precise results. Missing details in your arguments lead to point deductions.

Problem 1

Let X be a random sample of size n with X_1 distributed with PMF f_{X_1} given by

$$f_{X_1}(x|\theta) := \theta^x(1-\theta)^{1-x}$$

for $x \in \{0,1\}$ and $\theta \in [0,1/2]$.

(a) Show that the family $(f_{X_1}(\cdot|\theta), \theta \in [0,1/2])$ is an exponential family.

(b) Find a sufficient statistic for θ and show that it is sufficient.

(c) Derive a minimal sufficient statistic for θ and show that it is minimal.

(d) Derive a point estimator $\tilde{\theta}$ for θ with the method of moments.

(e) Compute the likelihood function of X and derive the maximum likelihood estimator $\hat{\theta}$.

(f) Compute the mean squared error of the method of moments estimator $\tilde{\theta}$.

(g) Show that the family $(f_{X_1}(\cdot|\theta), \theta \in [0,1/2])$ has a monotone likelihood ratio.

(26 points)

Problem 2

(a) Let f be a symmetric unimodal probability density function. Show that for a fixed value of $1-\alpha$, of all intervals $[a,b]$ that satisfy $\int_a^b f(x)\,dx = 1-\alpha$, a shortest is obtained by choosing a and b such that

$$\int_{-\infty}^{a} f(x)\,dx = \int_{b}^{\infty} f(x)\,dx = \frac{\alpha}{2}.$$

(b) Give a counterexample to show that the obtained shortest interval is not necessarily unique.

(c) Let X be a random sample of size n with $X_1 \sim \mathcal{N}(\mu, \sigma^2)$. For fixed $\alpha \in (0,1)$ compute a shortest $1-\alpha$ confidence interval for the parameter μ.

(13 points)

Problem 3
Test

$$H_0 : \theta = \theta_0 \qquad \text{versus} \qquad H_1 : \theta = \theta_1$$

with probability mass functions $f_X(\cdot|\theta_i)$, $i = 0, 1$, and rejection region R which satisfies that

- there exists $k \geq 0$ such that $x \in R$ if $f_X(x|\theta_1) > k f_X(x|\theta_0)$ and $x \in R^c$ if $f_X(x|\theta_1) < k f_X(x|\theta_0)$,
- $\alpha = P(X \in R)$.

(a) Show that any test that satisfies both assumptions is a uniformly most powerful level α test.

(b) Show further that if there exists a test that satisfies both assumptions for some $k > 0$, then every uniformly most powerful level α test is a size α test and every uniformly most powerful level α test satisfies the first assumption except perhaps on a set A satisfying $P(X \in A|\theta_0) = P(X \in A|\theta_1) = 0$.

(16 points)

Problem 4

(a) Give the definition of a consistent sequence of estimators of a parameter θ.

(b) Show that $(W_n, n \in \mathbb{N})$ is a consistent sequence of estimators of θ if for all $\theta \in \Theta$

   (a) $\lim_{n\to\infty} \mathsf{Var}[W_n] = 0$ and

   (b) $\lim_{n\to\infty} \mathsf{Bias}[W_n, \theta] = 0$.

(5 points)