CHALMERS | UNIVERSITY OF GOTHENBURG

Annika Lang

Statistical Inference Principles – Spring 2018

Assignment 1

1. Let $(X_i, i = 1, \ldots, n)$ be a random sample and X_1 has finite mean μ and variance σ^2. Define the two random variables

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

a) Show that $\bar{X}$ and S^2 are unbiased estimators for the mean and the variance of X.

b) Let M_X denote the *moment generating function* (MGF) of X given by

$$M_X(t) := \mathbb{E}[\exp(tX)], \quad t \in \mathbb{R}.$$

Show that the MGF $M_{\bar{X}}$ of $\bar{X}$ is given by

$$M_{\bar{X}}(t) = \left(M_X(n^{-1}t)\right)^n.$$

2. Let $\bar{X}_n$ be the sample mean. The goal of the task is to show in theory and simulation how the size of the sample influences the quality of the estimator in mean square.

a) Show that $\operatorname{Var}[\bar{X}_n] = n^{-1}\operatorname{Var}[X_1]$.

b) Write a program that returns for given mean μ, variance σ^2, and distribution, the approximate mean squared errors of the sample mean and sample variance for different sample sizes n. Plot your results (e.g., with a loglog plot) and find a rate of convergence. Compare different distributions and relate your findings to the results in a).

3. Let $(X_i, i = 1, \ldots, n)$ be a random sample from a Bernoulli(p) distribution, $p \in (0,1)$, i.e., the PMF satisfies $f(1) = p$ and $f(0) = 1 - p$.

a) Show that the family of Bernoulli(p) PMFs for $p \in (0,1)$ is an exponential family.

b) Show first that $T_1(X_1, \ldots, X_n) := \sum_{j=1}^{n} t_1(X_j)$ is binomial(n,p) distributed. Furthermore, show that the family of binomial(n,p) PMFs is an exponential family.

Please turn!

4. **a)** Let X be a random sample of odd size n and $X_1 \sim \mathcal{U}([0,1])$. Show that the sample median of X is an unbiased estimator of $\mathbb{E}[X_1]$.

b) Let X be a random sample of size 3 and X_1 Bernoulli(p)-distributed, i.e., $P(X_1 = 1) = p$ and $P(X_1 = 0) = 1 - p$. Find p such that the sample median becomes an unbiased estimator of $\mathbb{E}[X_1]$.

Deadline: Thursday, January 25, 2018, send an email before 14.30 with list of solved problems
Webpage: `http://www.math.chalmers.se/Stat/Grundutb/GU/MSF100/S18/`
Requirement: 75% of the exercises solved, two presentations in the exercise class