CHALMERS | UNIVERSITY OF GOTHENBURG

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Statistical Inference Principles – Spring 2018

Assignment 3

1. Let X be a random sample of size n with $X_1 \sim \mathcal{N}(\mu, 1)$. Derive the MLE of μ using the log likelihood function $l(\cdot|x) := \log L(\cdot|x)$ instead of maximizing $L(\cdot|x)$ directly.

2. Let X be a random sample of size n with PDF determined from

$$f_{X_1}(x|\theta) := \theta^x (1-\theta)^{1-x},$$

where $x \in \{0, 1\}$ and $\theta \in [0, 1/2]$.

a) Implement a method that generates samples of X.

b) Find the method of moments estimator $\hat{\theta}$ of θ and implement the method.

c) Find the MLE of θ and write an implementation. (*Hint:* Observe that $\theta \in [0, 1/2]$, so you might want to take care of the boundaries.)

d) Simulate samples of different sizes n and compare the results of both methods. What do you observe? Is one estimator performing better than the other? What do you declare to be "better" in that context?

3. Let X be a random sample of size n with $X_1 \sim \mathcal{N}(\theta, \sigma^2)$, and suppose that the prior distribution on θ is $\mathcal{N}(\mu, \tau^2)$. Assume that σ^2, μ, and τ^2 are known.

a) Find the joint PDF of $\bar{X}$ and θ.

b) Show that $m(\bar{x}|\sigma^2, \mu, \tau^2)$, the marginal distribution of $\bar{X}$, is $\mathcal{N}(\mu, \sigma^2/n + \tau^2)$.

c) Show that $\pi(\theta|\bar{x}, \sigma^2, \mu, \tau^2)$, the posterior distribution of θ, is normal with mean and variance

$$\mathbb{E}(\theta|\bar{x}) = \frac{\tau^2}{\tau^2 + \sigma^2/n}\bar{x} + \frac{\sigma^2/n}{\tau^2 + \sigma^2/n}\mu \quad \text{and} \quad \mathrm{Var}(\theta|\bar{x}) = \frac{\tau^2 \sigma^2/n}{\tau^2 + \sigma^2/n}.$$

4. Let $X := (X_i, i = 1, \ldots, m)$ and $Y := (Y_i, i = 1, \ldots, n)$ be random samples such that $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ is a random sample with $X_1 \sim \mathcal{N}(\mu, 1)$. Let X be the augmented data and Y be the incomplete data.

Please turn!

a) Compute all required quantities for the EM algorithm

b) Implement the EM algorithm for X and Y of arbitrary size that returns the sequence of estimated parameters $(\hat{\mu}^{(j)}, j = 1, \ldots)$. Test the algorithm by generating samples of Y and different initial guesses μ_0 and evaluate the performance of the algorithm using the resulting sequences.

Deadline: Thursday, February 8, 2018, send an email before 14.30 with a list of solved problems.
Webpage: `http://www.math.chalmers.se/Stat/Grundutb/GU/MSF100/S18/`
Requirement: 75% of the exercises solved, two presentations in the exercise class