

Tentamentsskrivning i MSF200/MVE330, 7.5 hp.

Tid: fredagen den 31 maj 2013 kl 8.30-12.30

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Hjälpmiddel: miniräknare, egen formelsamling på 4 A4 sidor (2 blad).

CTH: för "3" fordras 12 poäng, för "4" - 18 poäng, för "5" - 24 poäng.

GU: för "G" fordras 12 poäng, för "VG" - 20 poäng.

Inklusive eventuella bonus poäng.

1. (6 points) Define a process Y_n recursively by

$$Y_n = Z_n - 0.3 \cdot Y_{n-1}, \quad n \geq 1,$$

where Z_n are independent r.v. with zero means and unit variance.

- Under which conditions on Y_0 this process is weakly stationary? Compute its autocovariance function.
 - Find the best linear predictor \hat{Y}_{r+3} of Y_{r+3} given (Y_0, \dots, Y_r) .
 - What is the mean squared error of this prediction?
 - Give an example when this process is strongly stationary.
2. (4 points) Consider a Poisson process with parameter λ .
- Describe the Poisson process as a renewal process. Compute its renewal function.
 - Let $E(t)$ be the excess life at time t . Write down a renewal equation for $\mathbb{P}(E(t) > x)$. Solve this equation to find the distribution of $E(t)$.
 - What is the stationary distribution of the excess life?
3. (4 points) Let X_1, X_2, \dots be iid r.v. with finite mean μ , and let M be a stopping time with respect to the sequence X_n such that $\mathbb{E}(M) < \infty$. Then

$$\mathbb{E}(X_1 + \dots + X_M) = \mu \mathbb{E}(M).$$

- Give a detailed proof of the above formulated Wald's equation lemma.
 - Illustrate by an example that the statement may fail if M is not a stopping time.
4. (6 points) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of stochastic variables defined by

$$X_n = I_1 \sin(\phi n) + I_2 \cos(\phi n),$$

where I_1, I_2, ϕ are independent random variables such that $P(I_1 = 1) = P(I_1 = -1) = P(I_2 = 1) = P(I_2 = -1) = \frac{1}{2}$ and ϕ is uniformly distributed over $[-\pi, \pi]$.

- On the same graph draw three examples of trajectories of the process $\{X_n\}$. What is random in these trajectories?
- Verify that $\{X_n\}$ is a stationary process.
- Find the spectral density function of $\{X_n\}$.
- Find the spectral representation for the process $\{X_n\}$ in the following form

$$X_n = \int_0^\pi \cos(n\lambda) dU(\lambda) + \int_0^\pi \sin(n\lambda) dV(\lambda).$$

5. (4 points) Let random variables $\{X_n\}$ be iid random variables with $\mathbb{P}(X_n = 2) = \mathbb{P}(X_n = 0) = 1/2$.
- (a) Show that the product $S_n = X_1 \cdots X_n$ is a martingale.
 - (b) Show that $S_n \rightarrow 0$ in probability but not in mean.
 - (c) Does S_n converge a.s.?
6. (3 points) For a filtration (\mathcal{F}_n) let $B_n \in \mathcal{F}_n$. Put $X_n = 1_{B_1} + \dots + 1_{B_n}$.
- (a) Show that (X_n) is a submartingale.
 - (b) What is the Doob decomposition for X_n ?
7. (3 points) *Theorem. Let (Y_n, \mathcal{F}_n) be a submartingale and let T be a stopping time. Then $(Y_{T \wedge n}, \mathcal{F}_n)$ is a submartingale. If moreover, $\mathbb{E}|Y_n| < \infty$, then $(Y_n - Y_{T \wedge n}, \mathcal{F}_n)$ is also a submartingale.*
- (a) Referring to the above theorem carefully deduce the following statement. If (Y_n, \mathcal{F}_n) is a martingale and T is a stopping time, then $(Y_{T \wedge n}, \mathcal{F}_n)$ and $(Y_n - Y_{T \wedge n}, \mathcal{F}_n)$ are martingales.
 - (b) Illustrate this theorem with a simple example.

Partial answers and solutions are also welcome. Good luck!

Solutions

1. This is an example of AR(1) model $Y_n = \alpha Y_{n-1} + Z_n$. Observe that for $n \geq 0$, $m \geq 0$,

$$Y_{n+m} = Z_{n+m} + \alpha Z_{n+m-1} + \dots + \alpha^{n-1} Z_{m+1} + \alpha^n Y_m.$$

1a. Let Y_0 has mean μ and variance σ^2 . In the stationary case Y_n should also have mean μ and variance σ^2 . This leads to equations

$$\mu = \alpha^n \mu, \quad \sigma^2 = 1 + \alpha^2 + \dots + \alpha^{2n-2} + \alpha^{2n} \sigma^2,$$

which imply $\mu = 0$ and $\sigma^2 = \frac{1}{1-\alpha^2} \approx 1.0989$. Hence, using these values we compute for $n \geq 0$,

$$c(n) = \text{Cov}(Y_{n+m}, Y_m) = \mathbb{E}(Y_{n+m} Y_m) = \alpha^n \mathbb{E}(Y_m^2) = \frac{\alpha^n}{1-\alpha^2} = (-1)^n \frac{(0.3)^n}{0.91}.$$

1b. The best linear predictor $\hat{Y}_{r+k} = \sum_{j=0}^r a_j Y_{r-j}$ is found from the equations

$$\sum_{j=0}^r a_j c(|j-m|) = c(k+m), \quad 0 \leq m \leq r.$$

or

$$\sum_{j=0}^r a_j \alpha^{|j-m|} = \alpha^{k+m}, \quad 0 \leq m \leq r.$$

Intuition suggests seeking a solution of the form $\hat{Y}_{r+k} = a_0 Y_r$. Indeed, plugging $a_1 = \dots = a_r = 0$ we easily obtain $a_0 = \alpha^k$. Thus $\hat{Y}_{r+k} = (-0.3)^k Y_r$, and $\hat{Y}_{r+3} = -0.027 \cdot Y_r$.

1c. The mean square error is

$$\begin{aligned} \mathbb{E}((Y_{r+k} - \hat{Y}_{r+k})^2) &= \mathbb{E}((Y_{r+k} - \alpha^k Y_r)^2) = \mathbb{E}((Z_{r+k} + \alpha Z_{r+k-1} + \dots + \alpha^{k-1} Z_{r+1})^2) \\ &= \frac{1 - \alpha^{2k}}{1 - \alpha^2} = \frac{1 - (0.09)^k}{0.91} = 1.0981 \text{ for } k = 3. \end{aligned}$$

1d. If Y_0 has a normal distribution with mean zero and variance $\sigma^2 = \frac{1}{1-\alpha^2} \approx 1.0989$, then the process becomes Gaussian. For the Gaussian processes weak and strong stationarity are equivalent.

2a. Let $T_0 = 0$, $T_n = X_1 + \dots + X_n$, where X_i are iid exponential inter-arrival times with parameter λ . The Poisson process $N(t)$ is the renewal process giving the number of renewal events during the time interval $(0, t]$, so that

$$\{N(t) \geq n\} = \{T_n \leq t\}, \quad T_{N(t)} \leq t < T_{N(t)+1}.$$

Its renewal function $m(t) := \mathbb{E}(N(t))$ satisfies the renewal equation

$$m(t) = F(t) + \int_0^t m(t-u) dF(u), \quad F(t) = 1 - e^{-\lambda t}.$$

In terms of the Laplace-Stieltjes transforms $\hat{m}(\theta) := \int_0^\infty e^{-\theta t} dm(t)$ we get

$$\hat{m}(\theta) = \hat{F}(\theta) + \hat{m}(\theta) \hat{F}(\theta).$$

Since $\hat{F}(\theta) = \frac{\lambda}{\theta + \lambda}$, we obtain $\hat{m}(\theta) = \frac{\lambda}{\theta}$, implying $m(t) = \lambda t$.

2b. The excess life time $E(t) := T_{N(t)+1} - t$. The distribution of $E(t)$ is given by the following recursion for $b(t) := \mathbb{P}(E(t) > y)$

$$\begin{aligned} b(t) &= \mathbb{E}\left(\mathbb{E}(1_{\{E(t)>y\}}|X_1)\right) = \mathbb{E}\left(1_{\{E(t-X_1)>y\}}1_{\{X_1\leq t\}} + 1_{\{X_1>t+y\}}\right) \\ &= 1 - F(t+y) + \int_0^t b(t-x)dF(x). \end{aligned}$$

This renewal equation gives

$$\begin{aligned} \mathbb{P}(E(t) > y) &= 1 - F(t+y) + \int_0^t (1 - F(t+y-u))dm(u) \\ &= e^{-\lambda(t+y)} + \lambda \int_0^t e^{-\lambda(t+y-u)}du = e^{-\lambda(t+y)} + \lambda \int_y^{y+t} e^{-\lambda x}dx = e^{-\lambda y}. \end{aligned}$$

Thus $E(t)$ is geometrically distributed with parameter λ .

2c. From the previous calculation we see that the stationary distribution of $E(t)$ is geometric with parameter λ . This is in agreement with the general result, with μ standing for the mean inter-arrival time,

$$\lim_{t \rightarrow \infty} \mathbb{P}(E(t) \leq y) = \mu^{-1} \int_0^y (1 - F(x))dx = \lambda \int_0^y e^{-\lambda x}dx = 1 - e^{-\lambda y}.$$

3a. Let X_1, X_2, \dots be iid r.v. with finite mean μ , and let M be a stopping time with respect to the sequence X_n such that $\mathbb{E}(M) < \infty$. Then

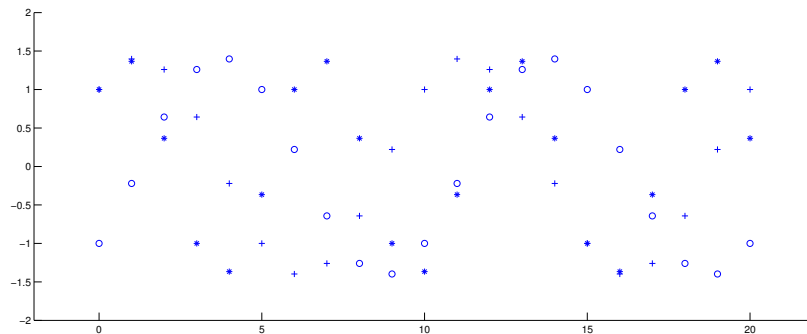
$$\mathbb{E}(X_1 + \dots + X_M) = \mu \mathbb{E}(M).$$

Proof. By dominated convergence

$$\begin{aligned} \mathbb{E}(X_1 + \dots + X_M) &= \mathbb{E}\left(\lim_{n \rightarrow \infty} \sum_{i=1}^n X_i 1_{\{M \geq i\}}\right) = \lim_{n \rightarrow \infty} \mathbb{E}\left(\sum_{i=1}^n X_i 1_{\{M \geq i\}}\right) \\ &= \sum_{i=1}^{\infty} \mathbb{E}(X_i) \mathbb{P}(M \geq i) = \mu \mathbb{E}(M). \end{aligned}$$

Here we used independence between $\{M \geq i\}$ and X_i , which follows from the fact that $\{M \geq i\}$ is the complimentary event to $\{M \leq i-1\} \in \sigma\{X_1, \dots, X_{i-1}\}$.

3c. Turn to the Poisson process and observe that $M = N(t)$ is not a stopping time and in this case $\mathbb{E}(T_{N(t)}) \neq \mu m(t)$. Indeed, for the Poisson process $\mu m(t) = t$ while $T_{N(t)} = t - C(t)$, where $C(t) > 0$ is the current lifetime.



4a. What looks on the figure as a cloud of stationary allocated points is produced by three sinusoids:

pluses: $\sin(\pi n/5) + \cos(\pi n/5)$,

circles: $\sin(\pi n/5) - \cos(\pi n/5)$,

stars: $\sin(\pi n/3) + \cos(\pi n/3)$.

The randomness is created by the random choice of the frequency and phase.

4b. The process

$$X_n = I_1 \sin(\phi n) + I_2 \cos(\phi n)$$

has zero means by independence and $\mathbb{E}(I_1) = \mathbb{E}(I_2) = 0$. Its autocovariancies are

$$\begin{aligned} \text{Cov}(X_n, X_m) &= \mathbb{E}(X_n X_m) = \mathbb{E}(\sin(\phi n) \sin(\phi m) + \cos(\phi n) \cos(\phi m)) \\ &= \mathbb{E}(\cos(\phi(n-m))) = \int_{-\pi}^{\pi} \cos(\phi(n-m)) d\phi = 1_{\{n=m\}}. \end{aligned}$$

Thus (X_n) is a weakly stationary sequence with mean zero, variance 1, and zero autocorrelation $\rho(n) = 0$ for $n \neq 0$.

4c. The spectral density g must satisfy $\rho(n) = \int_0^\pi \cos(n\lambda) g(\lambda) d\lambda$. Then it is the uniform density $g(\lambda) = \pi^{-1} 1_{\{\lambda \in [0, \pi]\}}$.

4d. In this case the spectral representation for the process $\{X_n\}$ is straightforward

$$X_n = I_1 \sin(\phi n) + I_2 \cos(\phi n) = \int_0^\pi \cos(n\lambda) dU(\lambda) + \int_0^\pi \sin(n\lambda) dV(\lambda),$$

where

$$U(\lambda) = I_1 1_{\{\lambda \geq \phi\}}, \quad V(\lambda) = I_2 1_{\{\lambda \geq \phi\}}.$$

Let us check the key properties of the random process $(U(\lambda), V(\lambda))$:

(i) $U(\lambda)$ and $V(\lambda)$ have zero means,

(ii) $U(\lambda_1)$ and $V(\lambda_2)$ are uncorrelated,

(iii) $\text{Var}(U(\lambda)) = \text{Var}(V(\lambda)) = \mathbb{E}(1_{\{\lambda \geq \phi\}}) = \lambda = G(\lambda)$,

(iv) increments of $U(\lambda)$ are uncorrelated, and increments of $V(\lambda)$ are uncorrelated.

The last property is obtained as follows: for $\lambda_1 < \lambda_2 < \lambda_3$,

$$\begin{aligned} \mathbb{E}(U(\lambda_3) - U(\lambda_2))(U(\lambda_2) - U(\lambda_1)) &= \mathbb{E}(1_{\{\lambda_3 \geq \phi\}} - 1_{\{\lambda_2 \geq \phi\}})(1_{\{\lambda_2 \geq \phi\}} - 1_{\{\lambda_1 \geq \phi\}}) \\ &= \mathbb{E}(1_{\{\lambda_2 < \phi \leq \lambda_3\}} 1_{\{\lambda_1 < \phi \leq \lambda_2\}}) = 0. \end{aligned}$$

5a. We have $S_n = 2^n$ with probability 2^{-n} and $S_n = 0$ with probability $1 - 2^{-n}$, so that $\mathbb{E}(S_n) = 1$. Clearly, $\mathbb{E}(S_{n+1}|S_n = 2^n) = 2^n$ and $\mathbb{E}(S_{n+1}|S_n = 0) = 0$, so that $\mathbb{E}(S_{n+1}|S_n) = S_n$.

5b. On one hand, $\mathbb{P}(|S_n| > \epsilon) = 2^{-n} \rightarrow 0$, so that $S_n \xrightarrow{P} 0$. On the other hand, $\mathbb{E}(S_n) = 1 \not\rightarrow 0$, so that S_n does not converge in mean.

5c. Put $A_n = \{S_n = 0\}$. We have $A_n \subset A_{n+1}$ and $A = \cup_n A_n = \{\omega : S_n(\omega) \rightarrow 0\}$. Since $\mathbb{P}(A) = \lim \mathbb{P}(A_n) = 1$, we conclude that $S_n \rightarrow 0$ almost surely.

6a. The nondecreasing sequence of nonnegative random variables $X_n(\omega)$ has finite means $a_n := \mathbb{E}(X_n) = \mathbb{P}(B_1) + \dots + \mathbb{P}(B_n)$. Therefore it is a submartingale.

6b. Doob's decomposition: $X_n = M_n + S_n$, where (M_n, \mathcal{F}_n) is a martingale and (S_n, \mathcal{F}_n) is an increasing predictable process (called the compensator of the submartingale). Here M and S are

computed as: $M_0 = 0$, $S_0 = 0$, and for $n \geq 0$

$$\begin{aligned} M_{n+1} - M_n &= X_{n+1} - \mathbb{E}(X_{n+1}|\mathcal{F}_n) = 1_{\{B_{n+1}\}} - \mathbb{P}(B_{n+1}), \\ S_{n+1} - S_n &= \mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n = \mathbb{P}(B_{n+1}). \end{aligned}$$

Thus $M_n = X_n - a_n$ and $S_n = a_n$.

7a. *Theorem 1. Let (Y_n, \mathcal{F}_n) be a submartingale and let T be a stopping time. Then $(Y_{T \wedge n}, \mathcal{F}_n)$ is a submartingale. If moreover, $\mathbb{E}|Y_n| < \infty$, then $(Y_n - Y_{T \wedge n}, \mathcal{F}_n)$ is also a submartingale.*

Changing the sign we easily get the next sister theorem.

Theorem 2. Let (Y_n, \mathcal{F}_n) be a supermartingale and let T be a stopping time. Then $(Y_{T \wedge n}, \mathcal{F}_n)$ is a supermartingale. If moreover, $\mathbb{E}|Y_n| < \infty$, then $(Y_n - Y_{T \wedge n}, \mathcal{F}_n)$ is also a supermartingale.

Since a martingale is both a submartingale and a supermartingale, these two theorems imply the following desired result.

Theorem 3. Let (Y_n, \mathcal{F}_n) be a martingale and let T be a stopping time. Then $(Y_{T \wedge n}, \mathcal{F}_n)$ is a martingale. If moreover, $\mathbb{E}|Y_n| < \infty$, then $(Y_n - Y_{T \wedge n}, \mathcal{F}_n)$ is also a martingale.

7b. Consider a symmetric simple random walk $Y_n = S_n$ and take T = the first hitting time of the level a . On the figure we show the trajectories of the three martingales in question.

