Tentamentsskrivning i MSF200/MVE330, 7.5 hp.

Tid: fredagen den 31 maj 2013 kl 8.30-12.30
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Hjälpmedel: miniräknare, egen formelsamling på 4 A4 sidor (2 blad).

CTH: för "3" fordras 12 poäng, för "4" - 18 poäng, för "5" - 24 poäng.
GU: för "G" fordras 12 poäng, för "VG" - 20 poäng.
Inklusive eventuella bonus poäng.

1. (6 points) Define a process Y_n recursively by

$$Y_n = Z_n - 0.3 \cdot Y_{n-1}, \quad n \geq 1,$$

where Z_n are independent r.v. with zero means and unit variance.

   (a) Under which conditions on Y_0 this process is weakly stationary? Compute its autocovariance function.

   (b) Find the best linear predictor $\hat{Y}_{r+3}$ of Y_{r+3} given $(Y_0, \ldots, Y_r)$.

   (c) What is the mean squared error of this prediction?

   (d) Give an example when this process is strongly stationary.

2. (4 points) Consider a Poisson process with parameter λ.

   (a) Describe the Poisson process as a renewal process. Compute its renewal function.

   (b) Let $E(t)$ be the excess life at time t. Write down a renewal equation for $\mathbb{P}(E(t) > x)$. Solve this equation to find the distribution of $E(t)$.

   (c) What is the stationary distribution of the excess life?

3. (4 points) Let $X_1, X_2, \ldots$ be iid r.v. with finite mean μ, and let M be a stopping time with respect to the sequence X_n such that $\mathbb{E}(M) < \infty$. Then

$$\mathbb{E}(X_1 + \ldots + X_M) = \mu \mathbb{E}(M).$$

   (a) Give a detailed proof of the above formulated Wald's equation lemma.

   (b) Illustrate by an example that the statement may fail if M is not a stopping time.

4. (6 points) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of stochastic variables defined by

$$X_n = I_1 \sin(\phi n) + I_2 \cos(\phi n),$$

where I_1, I_2, ϕ are independent random variables such that $P(I_1 = 1) = P(I_1 = -1) = P(I_2 = 1) = P(I_2 = -1) = \frac{1}{2}$ and ϕ is uniformly distributed over $[-\pi, \pi]$.

   (a) On the same graph draw three examples of trajectories of the process $\{X_n\}$. What is random in these trajectories?

   (b) Verify that $\{X_n\}$ is a stationary process.

   (c) Find the spectral density function of $\{X_n\}$.

   (d) Find the spectral representation for the process $\{X_n\}$ in the following form

$$X_n = \int_0^{\pi} \cos(n\lambda) dU(\lambda) + \int_0^{\pi} \sin(n\lambda) dV(\lambda).$$

5. (4 points) Let random variables $\{X_n\}$ be iid random variables with $\mathbb{P}(X_n = 2) = \mathbb{P}(X_n = 0) = 1/2$.

   (a) Show that the product $S_n = X_1 \cdots X_n$ is a martingale.

   (b) Show that $S_n \to 0$ in probability but not in mean.

   (c) Does S_n converge a.s.?

6. (3 points) For a filtration $(\mathcal{F}_n)$ let $B_n \in \mathcal{F}_n$. Put $X_n = 1_{B_1} + \ldots + 1_{B_n}$.

   (a) Show that (X_n) is a submartingale.

   (b) What is the Doob decomposition for X_n?

7. (3 points) *Theorem. Let* $(Y_n, \mathcal{F}_n)$ *be a submartingale and let* T *be a stopping time. Then* $(Y_{T\wedge n}, \mathcal{F}_n)$ *is a submartingale. If moreover,* $\mathbb{E}|Y_n| < \infty$*, then* $(Y_n - Y_{T\wedge n}, \mathcal{F}_n)$ *is also a submartingale.*

   (a) Referring to the above theorem carefully deduce the following statement. If $(Y_n, \mathcal{F}_n)$ is a martingale and T is a stopping time, then $(Y_{T\wedge n}, \mathcal{F}_n)$ and $(Y_n - Y_{T\wedge n}, \mathcal{F}_n)$ are martingales.

   (b) Illustrate this theorem with a simple example.

Partial answers and solutions are also welcome. Good luck!

Solutions

1. This is an example of AR(1) model $Y_n = \alpha Y_{n-1} + Z_n$. Observe that for $n \geq 0$, $m \geq 0$,

$$Y_{n+m} = Z_{n+m} + \alpha Z_{n+m-1} + \ldots + \alpha^{n-1} Z_{m+1} + \alpha^n Y_m.$$

1a. Let Y_0 has mean μ and variance σ^2. In the stationary case Y_n should also have mean μ and variance σ^2. This leads to equations

$$\mu = \alpha^n \mu, \quad \sigma^2 = 1 + \alpha^2 + \ldots + \alpha^{2n-2} + \alpha^{2n}\sigma^2,$$

which imply $\mu = 0$ and $\sigma^2 = \frac{1}{1-\alpha^2} \approx 1.0989$. Hence, using these values we compute for $n \geq 0$,

$$c(n) = \mathbb{C}ov(Y_{n+m}, Y_m) = \mathbb{E}(Y_{n+m}Y_m) = \alpha^n \mathbb{E}(Y_m^2) = \frac{\alpha^n}{1-\alpha^2} = (-1)^n \frac{(0.3)^n}{0.91}.$$

1b. The best linear predictor $\hat{Y}_{r+k} = \sum_{j=0}^{r} a_j Y_{r-j}$ is found from the equations

$$\sum_{j=0}^{r} a_j c(|j-m|) = c(k+m), \quad 0 \leq m \leq r.$$

or

$$\sum_{j=0}^{r} a_j \alpha^{|j-m|} = \alpha^{k+m}, \quad 0 \leq m \leq r.$$

Intuition suggests seeking a solution of the form $\hat{Y}_{r+k} = a_0 Y_r$. Indeed, plugging $a_1 = \ldots = a_r = 0$ we easily obtain $a_0 = \alpha^k$. Thus $\hat{Y}_{r+k} = (-0.3)^k Y_r$, and $\hat{Y}_{r+3} = -0.027 \cdot Y_r$.

1c. The mean square error is

$$\begin{aligned}\mathbb{E}\big((Y_{r+k} - \hat{Y}_{r+k})^2\big) &= \mathbb{E}\big((Y_{r+k} - \alpha^k Y_r)^2\big) = \mathbb{E}\big((Z_{r+k} + \alpha Z_{r+k-1} + \ldots + \alpha^{k-1} Z_{r+1})^2\big) \\ &= \frac{1-\alpha^{2k}}{1-\alpha^2} = \frac{1-(0.09)^k}{0.91} = 1.0981 \text{ for } k = 3.\end{aligned}$$

1d. If Y_0 has a normal distribution with mean zero and variance $\sigma^2 = \frac{1}{1-\alpha^2} \approx 1.0989$, then the process becomes Gaussian. For the Gaussian processes weak and strong stationarity are equivalent.

2a. Let $T_0 = 0$, $T_n = X_1 + \ldots + X_n$, where X_i are iid exponential inter-arrival times with parameter λ. The Poisson process $N(t)$ is the renewal process giving the number of renewal events during the time interval $(0, t]$, so that

$$\{N(t) \geq n\} = \{T_n \leq t\}, \quad T_{N(t)} \leq t < T_{N(t)+1}.$$

Its renewal function $m(t) := \mathbb{E}(N(t))$ satisfies the renewal equation

$$m(t) = F(t) + \int_0^t m(t-u)dF(u), \quad F(t) = 1 - e^{-\lambda t}.$$

In terms of the Laplace-Stieltjes transforms $\hat{m}(\theta) := \int_0^\infty e^{-\theta t} dm(t)$ we get

$$\hat{m}(\theta) = \hat{F}(\theta) + \hat{m}(\theta)\hat{F}(\theta).$$

Since $\hat{F}(\theta) = \frac{\lambda}{\theta+\lambda}$, we obtain $\hat{m}(\theta) = \frac{\lambda}{\theta}$, implying $m(t) = \lambda t$.

2b. The excess life time $E(t) := T_{N(t)+1} - t$. The distribution of $E(t)$ is given by the following recursion for $b(t) := \mathbb{P}(E(t) > y)$

$$b(t) = \mathbb{E}\Big(\mathbb{E}(1_{\{E(t)>y\}}|X_1)\Big) = \mathbb{E}\Big(1_{\{E(t-X_1)>y\}}1_{\{X_1\le t\}} + 1_{\{X_1>t+y\}}\Big)$$
$$= 1 - F(t+y) + \int_0^t b(t-x)dF(x).$$

This renewal equation gives

$$\mathbb{P}(E(t) > y) = 1 - F(t+y) + \int_0^t (1 - F(t+y-u))dm(u)$$
$$= e^{-\lambda(t+y)} + \lambda \int_0^t e^{-\lambda(t+y-u)}du = e^{-\lambda(t+y)} + \lambda \int_y^{y+t} e^{-\lambda x}dx = e^{-\lambda y}.$$

Thus $E(t)$ is exponentially distributed with parameter λ.

2c. From the previous calculation we see that the stationary distribution of $E(t)$ is geometric with parameter λ. This is in agreement with the general result, with μ standing for the mean inter-arrival time,

$$\lim_{t\to\infty} \mathbb{P}(E(t) \le y) = \mu^{-1} \int_0^y (1 - F(x))dx = \lambda \int_0^y e^{-\lambda x}dx = 1 - e^{-\lambda y}.$$

3a. Let $X_1, X_2, \ldots$ be iid r.v. with finite mean μ, and let M be a stopping time with respect to the sequence X_n such that $\mathbb{E}(M) < \infty$. Then

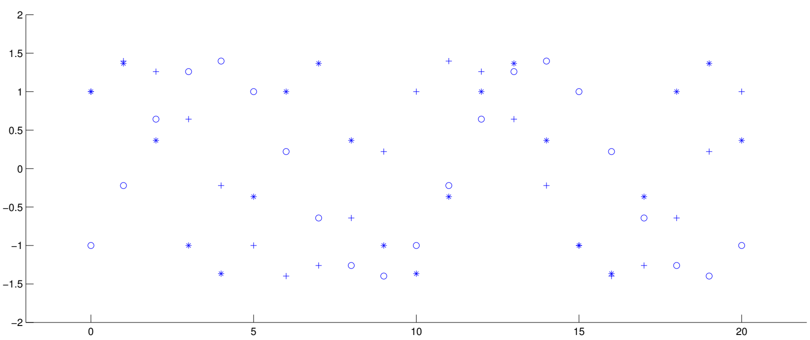
$$\mathbb{E}(X_1 + \ldots + X_M) = \mu\mathbb{E}(M).$$

Proof. By dominated convergence

$$\mathbb{E}(X_1 + \ldots + X_M) = \mathbb{E}\Big(\lim_{n\to\infty} \sum_{i=1}^n X_i 1_{\{M\ge i\}}\Big) = \lim_{n\to\infty} \mathbb{E}\Big(\sum_{i=1}^n X_i 1_{\{M\ge i\}}\Big)$$
$$= \sum_{i=1}^\infty \mathbb{E}(X_i)\mathbb{P}(M \ge i) = \mu\mathbb{E}(M).$$

Here we used independence between $\{M \ge i\}$ and X_i, which follows from the fact that $\{M \ge i\}$ is the complimentary event to $\{M \le i-1\} \in \sigma\{X_1, \ldots, X_{i-1}\}$.

3c. Turn to the Poisson process and observe that $M = N(t)$ is not a stopping time and in this case $\mathbb{E}(T_{N(t)}) \ne \mu m(t)$. Indeed, for the Poisson process $\mu m(t) = t$ while $T_{N(t)} = t - C(t)$, where $C(t) > 0$ is the current lifetime.



4a. What looks on the figure as a cloud of stationary allocated points is produced by three sinusoids:
pluses: $\sin(\pi n/5) + \cos(\pi n/5)$,
circles: $\sin(\pi n/5) - \cos(\pi n/5)$,
stars: $\sin(\pi n/3) + \cos(\pi n/3)$.
The randomness is created by the random choice of the frequency and phase.

4b. The process

$$X_n = I_1 \sin(\phi n) + I_2 \cos(\phi n)$$

has zero means by independence and $\mathbb{E}(I_1) = \mathbb{E}(I_2) = 0$. Its autocovariancies are

$$\begin{aligned}\mathbb{C}ov(X_n, X_m) &= \mathbb{E}(X_n X_m) = \mathbb{E}(\sin(\phi n)\sin(\phi m) + \cos(\phi n)\cos(\phi m)) \\ &= \mathbb{E}(\cos(\phi(n-m)) = \int_{-\pi}^{\pi} \cos(\phi(n-m))d\phi = 1_{\{n=m\}}.\end{aligned}$$

Thus (X_n) is a weakly stationary sequence with mean zero, variance 1, and zero autocorrelation $\rho(n) = 0$ for $n \neq 0$.

4c. The spectral density g must satisfy $\rho(n) = \int_0^\pi \cos(\lambda n) g(\lambda) d\lambda$. Then it is the uniform density $g(\lambda) = \pi^{-1} 1_{\{\lambda \in [0,\pi]\}}$.

4d. In this case the spectral representation for the process $\{X_n\}$ is straightforward

$$X_n = I_1 \sin(\phi n) + I_2 \cos(\phi n) = \int_0^\pi \cos(n\lambda) dU(\lambda) + \int_0^\pi \sin(n\lambda) dV(\lambda),$$

where

$$U(\lambda) = I_1 1_{\{\lambda \geq \phi\}}, \qquad V(\lambda) = I_2 1_{\{\lambda \geq \phi\}}.$$

Let us check the key properties of the random process $(U(\lambda), V(\lambda))$:
(i) $U(\lambda)$ and $V(\lambda)$ have zero means,
(ii) $U(\lambda_1)$ and $V(\lambda_2)$ are uncorrelated,
(iii) $\mathbb{V}ar(U(\lambda)) = \mathbb{V}ar(V(\lambda)) = \mathbb{E}(1_{\{\lambda \geq \phi\}}) = \lambda = G(\lambda)$,
(iv) increments of $U(\lambda)$ are uncorrelated, and increments of $V(\lambda)$ are uncorrelated.
The last property is obtained as follows: for $\lambda_1 < \lambda_2 < \lambda_3$,

$$\begin{aligned}\mathbb{E}(U(\lambda_3) - U(\lambda_2))(U(\lambda_2) - U(\lambda_1)) &= \mathbb{E}(1_{\{\lambda_3 \geq \phi\}} - 1_{\{\lambda_2 \geq \phi\}})(1_{\{\lambda_2 \geq \phi\}} - 1_{\{\lambda_1 \geq \phi\}}) \\ &= \mathbb{E}(1_{\{\lambda_2 < \phi \leq \lambda_3\}} 1_{\{\lambda_1 < \phi \leq \lambda_2\}}) = 0.\end{aligned}$$

5a. We have $S_n = 2^n$ with probability 2^{-n} and $S_n = 0$ with probability $1 - 2^{-n}$, so that $\mathbb{E}(S_n) = 1$. Clearly, $\mathbb{E}(S_{n+1}|S_n = 2^n) = 2^n$ and $\mathbb{E}(S_{n+1}|S_n = 0) = 0$, so that $\mathbb{E}(S_{n+1}|S_n) = S_n$.

5b. On one hand, $\mathbb{P}(|S_n| > \epsilon) = 2^{-n} \to 0$, so that $S_n \xrightarrow{\mathrm{P}} 0$. On the other hand, $\mathbb{E}(S_n) = 1 \nrightarrow 0$, so that S_n does not converge in mean.

5c. Put $A_n = \{S_n = 0\}$. We have $A_n \subset A_{n+1}$ and $A = \cup_n A_n = \{\omega : S_n(\omega) \to 0\}$. Since $\mathbb{P}(A) = \lim \mathbb{P}(A_n) = 1$, we conclude that $S_n \to 0$ almost surely.

6a. We verify three properties.
(i) Since $B_i \in \mathcal{F}_i \subset \mathcal{F}_n$, we conclude that X_n is measureable with respect to $\mathcal{F}_n$.
(ii) The random variables X_n have finite means $a_n := \mathbb{E}(X_n) = \mathbb{P}(B_1) + \ldots + \mathbb{P}(B_n)$.
(iii) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n + \mathbb{P}(B_{n+1}) \geq X_n$.

6b. Doob's decomposition: $X_n = M_n + S_n$, where $(M_n, \mathcal{F}_n)$ is a martingale and $(S_n, \mathcal{F}_n)$ is an increasing predictable process (called the compensator of the submartingale). Here M and S are computed as: $M_0 = 0$, $S_0 = 0$, and for $n \geq 0$

$$M_{n+1} - M_n = X_{n+1} - \mathbb{E}(X_{n+1}|\mathcal{F}_n) = 1_{\{B_{n+1}\}} - \mathbb{P}(B_{n+1}),$$
$$S_{n+1} - S_n = \mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n = \mathbb{P}(B_{n+1}).$$

Thus $M_n = X_n - a_n$ and $S_n = a_n$.

7a. *Theorem 1. Let $(Y_n, \mathcal{F}_n)$ be a submartingale and let T be a stopping time. Then $(Y_{T\wedge n}, \mathcal{F}_n)$ is a submartingale. If moreover, $\mathbb{E}|Y_n| < \infty$, then $(Y_n - Y_{T\wedge n}, \mathcal{F}_n)$ is also a submartingale.*

Changing the sign we easily get the next sister theorem.

Theorem 2. Let $(Y_n, \mathcal{F}_n)$ be a supermartingale and let T be a stopping time. Then $(Y_{T\wedge n}, \mathcal{F}_n)$ is a supermartingale. If moreover, $\mathbb{E}|Y_n| < \infty$, then $(Y_n - Y_{T\wedge n}, \mathcal{F}_n)$ is also a supermartingale.

Since a martingale is both a submartingale and a supermartingale, these two theorems imply the following desired result.

Theorem 3. Let $(Y_n, \mathcal{F}_n)$ be a martingale and let T be a stopping time. Then $(Y_{T\wedge n}, \mathcal{F}_n)$ is a martingale. If moreover, $\mathbb{E}|Y_n| < \infty$, then $(Y_n - Y_{T\wedge n}, \mathcal{F}_n)$ is also a martingale.

7b. Consider a symmetric simple random walk $Y_n = S_n$ and take $T =$ the first hitting time of the level a. On the figure we show the trajectories of the three martingales in question.