Tentamentsskrivning i MSF200/MVE330, 7.5 hp.

Tid: torsdagen den 3 juni 2014 kl 8.30-12.30
Examinator och jour: Serik Sagitov, tel. 772-5351, mob. 0736 907 613, rum H3026 i MV-huset.
Hjälpmedel: miniräknare, egen formelsamling på fyra A4 sidor (2 blad).

CTH: för “3” fordras 12 poäng, för “4” - 18 poäng, för “5” - 24 poäng.
GU: för “G” fordras 12 poäng, för “VG” - 20 poäng.

1. (5 points) Suppose $X_n \stackrel{L_2}{\to} X$ as $n \to \infty$.

   (a) Show that $X_n \stackrel{L_1}{\to} X$ and
   $$\mathbb{E}(X_n) \to \mathbb{E}(X).$$

   (b) Show that
   $$\mathrm{Var}(X_n) \to \mathrm{Var}(X).$$

   Reminder.
   Hölder's inequality: for $p, q > 1$ and $p^{-1} + q^{-1} = 1$,
   $$\mathbb{E}|XY| \le \big(\mathbb{E}|X^p|\big)^{1/p}\big(\mathbb{E}|Y^q|\big)^{1/q}.$$
   Minkowski's inequality: for $p \ge 1$,
   $$\big(\mathbb{E}|X+Y|^p\big)^{1/p} \le \big(\mathbb{E}|X^p|\big)^{1/p} + \big(\mathbb{E}|Y^p|\big)^{1/p}.$$

2. (5 points) Let (X_n) be independent random values with
   $$\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = 1/2,$$
   and consider the sequence of shifted numbers
   $$U_n = \sum_{i=1}^{\infty} X_{i+n} 2^{-i}, \quad n = 0, 1, \ldots$$
   For example, if $U_0 = 0.0110001\ldots$, then $U_1 = 0.110001\ldots$, $U_2 = 0.10001\ldots$, $U_3 = 0.0001\ldots$, and so on.

   (a) Show that each (U_n) is uniformly distributed on $[0,1]$.

   (b) Show that (U_n) is strongly stationary, and calculate its covariance function.

   (c) Knowing the past $(U_0, U_1, \ldots, U_r)$ find the best linear predictor $\hat{U}_{r+k}$ for the future value U_{r+k}. Discuss the result.

3. (5 points) Let the times between the events of a renewal process N be uniformly distributed on $[0,1]$.

   (a) Write down the renewal equation for the mean $m(t) = \mathbb{E}(N(t))$ and show that $m(t) = e^t - 1$ for $0 \le t \le 1$.

   (b) Write down the renewal equation for the second moment $m_2(t) = \mathbb{E}(N^2(t))$ and show that for $0 \le t \le 1$, the variance is

$$\mathrm{Var}(N(t)) = e^t(1 + 2t - e^t).$$

4. (5 points) Consider a G/G/1 queue. The arrivals of customers form a renewal process with the independent inter-arrival times X_n having an arbitrary common distribution. Let W_n be the waiting time of the n-th customer and S_n be its service time.

   (a) Show that the Lindley equation holds:

$$W_{n+1} = \max\{0, W_n + S_n - X_{n+1}\}.$$

   (b) Note that $U_n = S_n - X_{n+1}$ is a collection of iid r.v. Define an imbedded random walk by

$$\Sigma_0 = 0, \quad \Sigma_n = U_1 + \ldots + U_n.$$

   Show that

$$W_{n+1} \stackrel{d}{=} \max\{\Sigma_0, \ldots, \Sigma_n\}.$$

   (c) Let $F_n(x) = \mathbb{P}(W_n \le x)$ and $G(x) = \mathbb{P}(U_n \le x)$. Show that for $x \ge 0$,

$$F_{n+1}(x) = \int_{-\infty}^{x} F_n(x-y)dG(y).$$

5. (5 points) Let $\{Z_n\}_{n=0}^{\infty}$ be the Markov chain describing the size of a population of particles which reproduce themselves according to the following rules:

   - $Z_0 = 1$, so that the population stems from a single particle,
   - $Z_1 = 2$ with probability p and $Z_1 = 0$ with probability $1-p$,
   - $Z_{n+1} = X_{n,1} + \ldots + X_{n,Z_n}$, where $X_{n,1}, X_{n,2}, \ldots$ are independent copies of Z_1.

   In other words, each particle, independently of other coexisting particles, either produces no offspring or splits in two particles. The mean offspring number equals $m = 2p$.

   (a) Show that the ratio $\frac{Z_n}{m^n}$ is a martingale.

   (b) Let $p = 0.8$. Does $\frac{Z_n}{m^n}$ converge almost surely? Sketch a number of trajectories for $\frac{Z_n}{m^n}$ illustrating its typical asymptotic behavior.

   (c) Show that if $m = 1$, the martingale Z_n does not converge in L^2.

6. (5 points) Let $(Y_n, \mathcal{F}_n)$ be a submartingale and let T be a stopping time.

   (a) Give and discuss the definition of a stopping time T.

   (b) For $Z_n = Y_{T\wedge n}$, verify that

   $$Z_n = \sum_{i=0}^{n-1} Y_i 1_{\{T=i\}} + Y_n 1_{\{T\geq n\}}.$$

   Illustrate by drawing trajectories of (Y_n) and (Z_n).

   (c) Show that $(Y_{T\wedge n}, \mathcal{F}_n)$ is also a submartingale.

Partial answers and solutions are also welcome. Good luck!

Solutions summaries

1a. Using Hölder with $Y = 1$ we get

$$\mathbb{E}|X_n - X| \le \Big(\mathbb{E}(X_n - X)^2\Big)^{1/2}.$$

Thus $X_n \stackrel{L_2}{\to} X$ implies $X_n \stackrel{L_1}{\to} X$. Moreover, by Jensen,

$$|\mathbb{E}(X_n - X)| \le \mathbb{E}|X_n - X|,$$

which entails

$$\mathbb{E}(X_n) \to \mathbb{E}(X).$$

1b. To show that

$$\mathrm{Var}(X_n) \to \mathrm{Var}(X)$$

it is enough to prove that

$$\mathbb{E}(X_n^2) \to \mathbb{E}(X^2),$$

which is obtained using Minkowski twice:

$$\Big(\mathbb{E}(X_n^2)\Big)^{1/2} = \Big(\mathbb{E}(X_n - X + X)^2\Big)^{1/2} \le \Big(\mathbb{E}(X_n - X)^2\Big)^{1/2} + \Big(\mathbb{E}(X^2)\Big)^{1/2},$$
$$\Big(\mathbb{E}(X^2)\Big)^{1/2} = \Big(\mathbb{E}(X_n - X + X_n)^2\Big)^{1/2} \le \Big(\mathbb{E}(X_n - X)^2\Big)^{1/2} + \Big(\mathbb{E}(X_n^2)\Big)^{1/2}.$$

By L^2-convergence this yields the desired convergence

$$\limsup \mathbb{E}(X_n^2) \le \mathbb{E}(X^2) \le \liminf \mathbb{E}(X_n^2).$$

2a. By the iid property of (X_n) all (U_n) have the same distribution. Put

$$F_k(x) = \mathbb{P}\Big(\sum_{i=1}^{k} X_i 2^{-i} \le x\Big).$$

Clearly, $F_k(x)$ is the discrete uniform distribution over the set $\{m2^{-k}\}_{m=0}^{2^k-1}$. It remains to see that $F_k(x) \to x$ as $k \to \infty$ for any $x \in [0, 1]$.

2b. The sequence (U_n) is strongly stationary:

$$(U_n, \ldots, U_{n+k}) \stackrel{d}{=} (U_0, \ldots, U_k)$$

since

$$(X_{n+1}, X_{n+2}, \ldots) \stackrel{d}{=} (X_1, X_2, \ldots).$$

Its autocovariance function $c(k)$ for $k \leq 0$, is

$$c(k) = \mathbb{E}(U_0 U_k) - 1/4 = \mathbb{E}(\{\sum_{i=1}^{k} X_i 2^{-i} + U_k 2^{-k}\} U_k) - 1/4$$
$$= 2^{-1}\mathbb{E}(\sum_{i=1}^{k} X_i 2^{-i}) + 2^{-k}\mathbb{E}(U_k^2) - 1/4 = \frac{2^{-k}}{12}.$$

2c. Knowing the past $(U_0, U_1, \ldots, U_r)$ find best linear predictor for the future value U_{r+k} has the form

$$\hat{U}_{r+k} = a_0 U_r + \ldots + a_r U_0 + (1 - a_0 - \ldots - a_r) \cdot \frac{1}{2},$$

where the coefficients a_i are found from the linear equations

$$\begin{aligned} a_0 c(0) + a_1 c(1) + \ldots + a_r c(r) &= c(k), \\ a_0 c(1) + a_1 c(0) + \ldots + a_r c(r-1) &= c(k+1), \ldots \\ a_0 c(r) + a_1 c(r-1) + \ldots + a_r c(0) &= c(k+r), \end{aligned}$$

that is

$$\begin{aligned} a_0 + a_1 2^{-1} + \ldots + a_r 2^{-r} &= 2^{-k}, \\ a_0 2^{-1} + a_1 + \ldots + a_r 2^{-r+1} &= 2^{-k-1}, \ldots \\ a_0 2^{-r} + a_1 2^{-r+1} + \ldots + a_r &= 2^{-k-r}. \end{aligned}$$

It is easy to find that $a_0 = 2^{-k}$ and $a_1 = \ldots = a_r = 0$ satisfy this system. So that

$$\hat{U}_{r+k} = 2^{-k} U_r + 2^{-1} - 2^{-k-1}.$$

Its mean square error error is

$$\mathbb{E}(\hat{U}_{r+k} - U_{r+k})^2 = \mathbb{E}(2^{-k}(U_r - 2^{-1}) - (U_{r+k} - 2^{-1}))^2 = \frac{2^{-2k} + 1}{12} - 2 \cdot 2^{-k} \cdot \frac{2^{-k}}{12} = \frac{1 - 2^{-2k}}{12}.$$

Observe that the best (not necessarily linear) prediction has zero error, since knowing U_0 you know all the following U_n.

3a. Let the times between the events of a renewal process N be uniformly distributed on $[0, 1]$ so that $F(t) = t \cdot 1_{\{0 \leq t \leq 1\}}$. The renewal property gives

$$N(t) = 1_{\{X_1 \leq t\}}(1 + \tilde{N}(t - X_1)).$$

Taking expectations we see that $m(t) = \mathbb{E}N(t)$ satisfies the renewal equation

$$m(t) = F(t) + \int_0^t m(t - u) dF(u).$$

For $0 \le t \le 1$, we have

$$m(t) = t + \int_0^t m(u)du,$$

and therefore $m'(t) = 1 + m(t)$ with $m(0) = 0$. Thus $m(t) = e^t - 1$ for $0 \le t \le 1$.

3b. From

$$N(t) = 1_{\{X_1 \le t\}}(1 + \tilde{N}(t - X_1))$$

we get

$$N^2(t) = 1_{\{X_1 \le t\}}(1 + 2\tilde{N}(t - X_1) + \tilde{N}^2(t - X_1))$$

Taking expectations we obtain the renewal equation for the second moment $m_2(t) = \mathbb{E}(N^2(t))$

$$m_2(t) = A(t) + \int_0^t m_2(t - u)dF(u),$$

where

$$A(t) = \int_0^t (1 + 2m(t - u))dF(u).$$

Using the renewal function $U(t) = 1 + m(t)$ we find the solution of this renewal equation as

$$m_2(t) = \int_0^t A(t - u)dU(u).$$

For $0 \le t \le 1$, we have $U(t) = e^t$ and

$$A(t) = \int_0^t (2e^u - 1)du = 2e^t - 2 - t,$$

so that

$$\begin{aligned} m_2(t) &= 2e^t - 2 - t + \int_0^t (2e^{t-u} - 2 - t + u)e^u du \\ &= te^t + \int_0^t ue^u du = 1 - e^t + 2te^t. \end{aligned}$$

Thus for $0 \le t \le 1$, the variance is

$$\mathrm{Var}(N(t)) = 1 - e^t + 2te^t - (e^t - 1)^2 = e^t + 2te^t - e^{2t} = e^t(1 + 2t - e^t).$$

4. See the Lecture Notes, the section on G/G/1 queues.

5a. The branching process $\{Z_n\}_{n=0}^\infty$ has the mean offspring number equals $m = 2p$. Put $Y_n = \frac{Z_n}{m^n}$. Clearly, $0 \le Z_n \le 2^n$ so that $0 \le Y_n \le p^{-n}$. Using the branching property

$$Z_{n+1} = X_{n,1} + \ldots + X_{n,Z_n},$$

we find

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = m^{-n-1} m Z_n = Y_n.$$

Thus the ratio $\frac{Z_n}{m^n}$ is a martingale.

5b. If $p = 0.8$, then $m = 1.6$. The non-negative martingale $\frac{Z_n}{m^n}$ converges almost surely. There are two types of trajectories for $\frac{Z_n}{m^n}$ in the supercritical case ($m > 1$). Some trajectories go to zero (extinction), and some trajectories converge to (different) non-zero values.

5c. In the critical case $m = 1$, the branching process Z_n always converge to zero. Since $\mathbb{E}(Z_n) = 1$ the martingale (Z_n) does not converge in L^1 and therefore, does not converge in L^2.

6. See the Lecture Notes, the section on Bounded stopping times and Optional sampling theorem.