Tentamentsskrivning i MSF200/MVE330, 7.5 hp.

Tid: fredagen den 3 juni 2016 kl 8.30-12.30
Examinator och jour: Serik Sagitov, tel. 031-7725351, rum H3026 i MV-huset.
Hjälpmedel: miniräknare, egen formelsamling på fyra A4 sidor (2 blad).

CTH: för "3" fordras 12 poäng, för "4" - 18 poäng, för "5" - 24 poäng.
GU: för "G" fordras 12 poäng, för "VG" - 20 poäng.

1. (5 points) Let $X_1, X_2, \ldots$ be exponentially distributed inter-arrival times with different parameters: $EX_i = \frac{1}{i(i+1)}$. The arrival time of the n-th customer is then

$$S_n = X_1 + \ldots + X_n.$$

   (a) Show that (S_n) is a submartingale. Prove that there is a finite r.v. Y such that $S_n \to Y$ as $n \to \infty$ almost surely.

   (b) Show that $(Y - S_n - \frac{1}{n+1})$ is a backward martingale. What does this imply concerning the limit behaviour of $Y_n = Y - S_n$ as $n \to \infty$?

   (c) Prove that $nY_n \to 1$ almost surely.
   Hint. Use as known the following fact $E(nY_n - 1)^4 = O(n^{-2})$ as $n \to \infty$.

2. (5 points) Recall the renewal theorem claiming that if X_1 is not arithmetic, then for any positive h

$$U(t+h) - U(t) \to \mu^{-1}h, \quad t \to \infty.$$

   (a) Prove the key renewal theorem

$$\int_0^t g(t-u)dU(u) \to \mu^{-1} \int_0^\infty g(u)du, \quad t \to \infty,$$

   for a step function

$$g(t) = c_1 1_{\{a_1 \le t \le b_1\}} + \ldots + c_n 1_{\{a_n \le t \le b_n\}}.$$

   (b) Using the key renewal theorem find the stationary distribution for the excess lifetime $E(t)$.

3. (5 points) Let $X_1, X_2, \ldots$ be iid random variables with distribution

$$\mathrm{P}(X = 1) = \mathrm{P}(X = 2) = 0.25, \quad \mathrm{P}(X = 3) = 0.5.$$

   Let (N_n) be the discrete-time renewal process with (X_n) being the inter-arrival times.

   (a) Draw three plausible trajectories of the random process (N_n). Using these illustrate the concept of excess and current lifetimes (E_n, C_n).

(b) Find the distribution of C_1 and then of E_2.

(c) In what sense (N_n) is a submartingale?

4. (5 points) Consider a G/G/1 queue with the inter-arrival times of customers having the distribution

$$\mathrm{P}(X = 1) = \mathrm{P}(X = 2) = 0.25, \quad \mathrm{P}(X = 3) = 0.5.$$

Let the service times be equal 2 with probability 1.

(a) Show that the corresponding Wiener-Hopf equation

$$F(x) = \mathrm{E}(F(x - U); U \leq x),$$

where U is the difference between the service time and inter-arrival time for a customer, is equivalent to

$$3F(x) = F(x-1) + 2F(x+1), \quad x \geq 1, \qquad F(1) = (3/2)F(0).$$

(b) Check that $F(x) = F(0)(2 - 2^{-x})$ and give its interpretation in terms of the embedded random walk.

(c) Compute the traffic intensity ρ. Find the probability $(1-\eta)$ that the equilibrium waiting time for a customer is zero. Hint: show that $\eta = 1/3 + (2/3)\eta^2$.

5. (5 points)

(a) Give the definition of a stopping time with respect to a filtration $(\mathcal{F}_n)$.

(b) Prove: if $T \leq S$ are two stopping times, then $\mathcal{F}_T \subset \mathcal{F}_S$.

(c) What does the optional sampling theorem tell us?

6. (5 points)

(a) A Gaussian random process is strongly stationary if and only if it is weakly stationary. Explain.

(b) Give an example of a strongly stationary sequence (X_n) with the marginal distribution (4), such that $\frac{X_1+\ldots+X_n}{n}$ does not converge to the marginal mean $\mu = 9/4$.

(c) Let (X_n) be iid r.v. with Bernoulli (p) distribution. Put $W_n = W_0 + Y_n/\sqrt{npq}$, where the r.v.

$$Y_n = X_1 + \ldots + X_n - np$$

has mean 0 and variance npq. Can we choose a r.v. W_0, which is independent of (X_n), in such a way that $(W_n)_{n\geq 0}$ becomes a weakly stationary process? Sketch a typical trajectory.

Partial answers and solutions are also welcome. Good luck!

Solutions summaries

1a. Observe that for all n,

$$\mathrm{E}S_n = (1 - 1/2) + (1/2 - 1/3) + \ldots + (1/n - 1/(n+1)) = 1 - 1/(n+1).$$

The process (S_n) is a submartingale with respect to the filtration $(\mathcal{F}_n)$ generated by the r.v. $(X_1, \ldots, X_n)$ since

- S_n is measurable with respect to the sigma-algebra $\mathcal{F}_n$,
- S_n are positive with $\mathrm{E}S_n < 1$,
- $\mathrm{E}(S_{n+1}|\mathcal{F}_n) = S_n + \frac{1}{(n+1)(n+2)} > S_n$.

Finally, we know that

> "Any martingale, submartingale or supermartingale $(Y_n, \mathcal{F}_n)$ satisfying $\sup_n \mathrm{E}|Y_n| \leq M$ converges almost surely to a r.v. with a finite mean."

Thus

$$S_n \to Y = \sum_{i \geq 1} X_i, \quad n \to \infty,$$

almost surely, and $\mathrm{E}Y = 1$.

1b. Consider

$$Y_n = Y - S_n = \sum_{i \geq n+1} X_i,$$

and observe that $\mathrm{E}Y_n = \frac{1}{n+1}$. The sequence $(Y_n - \frac{1}{n+1})$ is a backward martingale with respect to

$$(\mathcal{G}_n), \quad \mathcal{G}_n = \sigma\{X_{n+1}, X_{n+2}, \ldots\},$$

since

$$\mathrm{E}(Y_n|\mathcal{G}_{n+1}) = \mathrm{E}X_{n+1} + \sum_{i \geq n+2} X_i = 1/(n+1) + Y_{n+1} - 1/(n+2).$$

From the backward martingale property it follows that $Y_n - \frac{1}{n+1} \to 0$, or equvalently $Y_n \to 0$, almost surely and in mean.

1c. Using the Markov inequality

$$\mathrm{P}(|nY_n - 1| > \epsilon) \leq \epsilon^{-4}\mathrm{E}(nY_n - 1)^4,$$

we see that

$$\sum_{n \geq 1} \mathrm{P}(|nY_n - 1| > \epsilon) < \infty.$$

By the the first Borel-Cantelli lemma, the events $\{|nY_n - 1| > \epsilon\}$ happen finitely often, implying that for a given realisation, $|nY_n - 1| \le \epsilon$ for all sufficiently large n, implying the stated almost sure convergence.

3b. The current lifetime $C_1 = 1_{\{X_1>1\}}$ can take values 0 or 1 with probabilities

$$\mathrm{P}(C_1 = 0) = \mathrm{P}(X_1 = 1) = 0.25, \quad \mathrm{P}(C_1 = 1) = 0.75.$$

The excess lifetime E_2 can take values 1, 2, or 3 with probabilities

$$\begin{aligned}
\mathrm{P}(E_2 = 1) &= \mathrm{P}(X_1 = 3) + \mathrm{P}(X_1 = 2, X_2 = 1) + \mathrm{P}(X_1 = 1, X_2 = 2) \\
&\quad + \mathrm{P}(X_1 = 1, X_2 = 1, X_3 = 1) = 1/2 + 1/16 + 1/16 + 1/64 = 41/64, \\
\mathrm{P}(E_2 = 2) &= \mathrm{P}(X_1 = 2, X_2 = 2) + \mathrm{P}(X_1 = 1, X_2 = 3) + \mathrm{P}(X_1 = 1, X_2 = 1, X_3 = 2) \\
&= 1/16 + 1/8 + 1/64 = 13/64, \\
\mathrm{P}(E_2 = 3) &= \mathrm{P}(X_1 = 2, X_2 = 3) + \mathrm{P}(X_1 = 1, X_2 = 1, X_3 = 3) = 1/8 + 1/32 = 10/64.
\end{aligned}$$

4a. From

$$\mathrm{P}(U = 1) = \mathrm{P}(U = 0) = 0.25, \quad \mathrm{P}(U = -1) = 0.5,$$

we can write the Wiener-Hopf equation as

$$F(x) = (1/4)F(x-1) + (1/4)F(x) + (1/2)F(x+1),$$

or

$$3F(x) = F(x-1) + 2F(x+1).$$

The function F is the distribution function for the waiting time of a customer in a stationary regime. Thus $F(x) = 0$ for negative x. It follows from the previous relation with $x = 0$ that $3F(0) = 2F(1)$ so that $F(1) = (3/2)F(0)$.

4b. We have to check that $F(x) = F(0)(2 - 2^{-x})$ satisfies the recursion

$$3F(x) = F(x-1) + 2F(x+1), \quad x \le 1.$$

Indeed

$$3F(x) = 3F(0)(2-2^{-x}) = F(0)(2-2\cdot 2^{-x}) + 2F(0)(2-(1/2)2^{-x}) = F(x-1) + 2F(x+1).$$

In terms of the embedded random walk $F(x)$ is the probability that the random walk never goes above the level x.

4c. The traffic intensity $\rho = \frac{2}{9/4} = 8/9$ is less than 1, implying that in the long run we will get a stationary regime.

5b. Let $T \leq S$ are two stopping times with respect to filtration $(\mathcal{F}_n)$. For an event $A \in \mathcal{F}_T$, so that

$$A \cap \{T = n\} \in \mathcal{F}_n, \quad n \geq 0,$$

we have to show $A \in \mathcal{F}_S$, so that

$$A \cap \{S = n\} \in \mathcal{F}_n, \quad n \geq 0.$$

The latter is obtained from the decomposition

$$A \cap \{S = n\} = A \cap \{S = n\} \cap \{T \leq n\} = \cup_{i=1}^{n} A \cap (\{S = n\} \cap \{T = i\}),$$

where for each $i \leq n$,

$$(A \cap \{T = i\}) \cap \{S = n\} \in \mathcal{F}_n.$$

6c. Eventhough the mean $\mathrm{E}W_n = 0$ for all n, the variance can not be constant since

$$\mathrm{Var}W_n = \mathrm{Var}W_0 + 1 > \mathrm{Var}W_0.$$

So the sequence $(W_n)_{n\geq 0}$ can not be stationary.