Tentamentsskrivning i MSF200/MVE330, 7.5 hp.

Tid: fredagen den 1 juni 2018 kl 8.30-12.30
Examinator och jour: Serik Sagitov, tel. 031-7725351, rum H3026 i MV-huset.
Hjälpmedel: miniräknare, egen kurs sammanfattning på fyra A4 sidor (dvs 2 blad).

Chalmers: för "3" fordras 12 poäng, för "4" - 18 poäng, för "5" - 24 poäng.
GU: för "G" fordras 12 poäng, för "VG" - 20 poäng. Inklusive eventuella bonus poäng.

> **Reminder.** Attach your digest report to the exam solutions. If the report is generated by Latex and appropriately summarises the course, you may get bonus point(s) for it.

1. (5 points) Define a sequence of random variables $X_2, X_3, \dots$ on the probability space $\{[0,1], \mathcal{B}_{[0,1]}, \mathrm{P}\}$, where $\mathrm{P}(d\omega) = d\omega$ is the Lebesgue measure, by

$$X_n(\omega) = n \cdot 1_{\{n^{-1}<\omega<2n^{-1}\}} - (n+1) \cdot 1_{\{(n+1)^{-1}<\omega<2(n+1)^{-1}\}}.$$

   (a) Show that $\mathrm{E}(X_n) = 0$. Check if (X_n) is a weakly stationary sequence.

   (b) Sketch a typical trajectory of (X_n) and explain how you do it. Does X_0 converge almost surely as $n \to \infty$?

   (c) Does X_n converge in mean as $n \to \infty$?

2. (5 points) Consider a Poisson process $\{N(t), t \geq 0\}$ with parameter $\lambda = 1$. Distinguishing between short and long inter-arrival times (X_n) put

$$N(t) = N_s(t) + N_l(t),$$

   where $N_s(t)$ is the number of X_n observed in the time interval $[0, t]$ which were smaller than $\ln 2$. Take the difference

$$D(t) = N_s(t) - N_l(t).$$

   (a) Sketch a typical trajectory of $D(t)$. What happens with it in the long run?

   (b) Referring to one of the results presented in this course, describe the long term behaviour of of $D(t)/t$.

3. (5 points) Let $(X_n, n \geq 1)$ be a sequence of independent random values with

$$\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = \tfrac{1}{2}) = \tfrac{1}{2},$$

   and define (U_n) as an autoregression AR(1) process

$$U_{n+1} = \tfrac{1}{2}U_n + X_{n+1}, \quad n \geq 1,$$

   where U_0 has a uniform distribution over the unit interval $[0, 1]$ which is independnt of the sequence of innovations (X_n).

(a) Sketch a typical trajectory of (U_n). From your drawings, would you expect that the autocorrelations are positive, negative, or zero?

(b) Compute the distribution of U_n.

(c) Calculate the autocorrelation function of (U_n).

(d) Is (U_n) a stationary random process? If not, why? If yes, in what sense?

4. (5 points) Consider a G/G/1 queue, with both the inter-arrival times of customers and service times having gamma distributions. The former has mean and variance $(\frac{\alpha}{\lambda}, \frac{\alpha}{\lambda^2})$, and the latter has mean and variance $(\frac{\alpha}{\mu}, \frac{\alpha}{\mu^2})$.

   (a) Find the traffic intensity parameter. Which set of parameters (α, λ, μ) characterises the light traffic regime of the model? Explain.

   (b) How would you generate the trajectories of the queue length for this model for a given set of parameters $(\alpha, \lambda, \mu) = (2, 3, 3)$? Indicate major steps of your algorithm.

   (c) How the trajectories generated by the algorithm (b) would look like in the long run? Explain referring to the properties of an associated random walk.

   (d) How the light traffic regime is identified in the general setting of the Little Law?

5. (5 points) Consider a random walk $S_n = 1 + X_1 + \ldots + X_n$ with independent jump sizes having a common distribution

$$\mathrm{P}(X_i = 1) = \mathrm{P}(X_i = -1) = p, \quad \mathrm{P}(X_i = 0) = 1 - 2p,$$

   for some $p \in (0, \frac{1}{2}]$. Put $Y_n = 2^{S_n}$.

   (a) Find for which values of p the process $(Y_n, n \geq 0)$ is a submartingale, is a martingale, is a supermartingale. Specify the underlying filtration.

   (b) In the case $(Y_n, n \geq 0)$ is a submartingale, compute its compensator.

6. (5 points) Let T be a stopping time with respect to a filtration $(\mathcal{F}_n, n \geq 0)$. For any $n \geq 0$, let Y_n be $\mathcal{F}_n$-measurable. Put

$$Z_n = Y_n + \sum_{i=1}^{n} (Y_i - Y_n) 1_{\{T=i\}}, \quad n \geq 0.$$

   (a) Show that

$$Z_n = Y_{T \wedge n}.$$

   Draw several plausible trajectories of (Y_n) and then on top of them, the corresponding trajectories of (Z_n).

   (b) What interpretation has the process (Z_n) in the gambling setting, when (Y_n) is assumed to be a supermartingale? Justify the requirement of T being a stopping time.

(c) Prove that

$$\mathrm{E}(Z_{n+1}|\mathcal{F}_n) = \mathrm{E}(Y_{n+1}|\mathcal{F}_n)1_{\{T\geq n+1\}} + Y_T 1_{\{T\leq n\}}.$$

(d) Derive from (b) that given (Y_n) is a supermartingale, (Z_n) is also a supermartingale.

Partial answers and solutions are also welcome. Good luck!

Solutions summaries

1a. We have $\mathrm{E}X_n = 0$, and therefore,

$$\begin{aligned}\mathrm{Cov}(X_n, X_{n+k}) = \mathrm{E}(X_n X_{n+k}) &= n(n+k)\mathrm{P}\{(\tfrac{1}{n},\tfrac{2}{n})\cap(\tfrac{1}{n+k},\tfrac{2}{n+k})\}\\ &-(n+1)(n+k)\mathrm{P}\{(\tfrac{1}{n+1},\tfrac{2}{n+1})\cap(\tfrac{1}{n+k},\tfrac{2}{n+k})\}\\ &-n(n+k+1)\mathrm{P}\{(\tfrac{1}{n},\tfrac{2}{n})\cap(\tfrac{1}{n+k+1},\tfrac{2}{n+k+1})\}\\ &+(n+1)(n+k+1)\mathrm{P}\{(\tfrac{1}{n+1},\tfrac{2}{n+1})\cap(\tfrac{1}{n+k+1},\tfrac{2}{n+k+1})\}.\end{aligned}$$

This yields

$$\begin{aligned}\mathrm{Var}X_n &= n^2(\tfrac{2}{n}-\tfrac{1}{n}) - 2n(n+1)(\tfrac{2}{n+1}-\tfrac{1}{n}) + (n+1)^2(\tfrac{2}{n+1}-\tfrac{1}{n+1})\\ &= 3,\end{aligned}$$

and for $k \geq 1$, $n \geq 2$,

$$\begin{aligned}\mathrm{Cov}(X_n, X_{n+k}) &= n(n+k)(\tfrac{2}{n+k}-\tfrac{1}{n})_+\\ &-(n+1)(n+k)(\tfrac{2}{n+k}-\tfrac{1}{n+1})_+\\ &-n(n+k+1)(\tfrac{2}{n+k+1}-\tfrac{1}{n})_+\\ &+(n+1)(n+k+1)(\tfrac{2}{n+k+1}-\tfrac{1}{n+1})_+\\ &=(n-k)_+-(n-k+2)_+-(n-k-1)_++(n-k+1)_+\\ &=-1_{\{n=k\}}-1_{\{n=k-1\}}.\end{aligned}$$

Not stationary.

1c. No convergence in mean, since

$$\mathrm{E}|X_n| = n(\tfrac{2}{n}-\tfrac{1}{n}) + (n+1)(\tfrac{2}{n+1}-\tfrac{1}{n+1}) = 2.$$

3b. Using the total probability law check that the distribution of U_1 is uniform over [0,1]. Thus all U_n have the same distribution.

3c. We have

$$\begin{aligned}\mathrm{E}(U_nU_{n+k}) &= \mathrm{E}(U_n(\tfrac{1}{2}U_{n+k-1}+X_{n+k})) = \tfrac{1}{2}\mathrm{E}(U_nU_{n+k-1})+\tfrac{1}{8}\\ &= \tfrac{1}{2^k}\mathrm{E}(U_n^2)+\tfrac{1}{8}(1+\tfrac{1}{2}+\ldots+\tfrac{1}{2^{k-1}}) = \tfrac{1}{4}+\tfrac{1}{12\cdot 2^k},\end{aligned}$$

which yields

$$\mathrm{Cov}(U_n, U_{n+k}) = \mathrm{E}(U_nU_{n+k}) - \tfrac{1}{4} = \tfrac{1}{12\cdot 2^k}.$$

We conclude that the autocorrelation function is stationary

$$\rho(k) = \tfrac{1}{2^k}.$$

3d. The process is a stationary Markov process and therefore is stationary in the strong sense.

4a. From

$$P(U = 1) = P(U = 0) = 0.25, \quad P(U = -1) = 0.5,$$

we can write the Wiener-Hopf equation as

$$F(x) = (1/4)F(x - 1) + (1/4)F(x) + (1/2)F(x + 1),$$

or

$$3F(x) = F(x - 1) + 2F(x + 1).$$

The function F is the distribution function for the waiting time of a customer in a stationary regime. Thus $F(x) = 0$ for negative x. It follows from the previous relation with $x = 0$ that $3F(0) = 2F(1)$ so that $F(1) = (3/2)F(0)$.

4b. We have to check that $F(x) = F(0)(2 - 2^{-x})$ satisfies the recursion

$$3F(x) = F(x - 1) + 2F(x + 1), \quad x \leq 1.$$

Indeed

$$3F(x) = 3F(0)(2-2^{-x}) = F(0)(2-2\cdot 2^{-x})+2F(0)(2-(1/2)2^{-x}) = F(x-1)+2F(x+1).$$

In terms of the embedded random walk $F(x)$ is the probability that the random walk never goes above the level x.

5a. From the conditional expectation formula

$$E(Y_{n+1}|\mathcal{F}_n) = E(2^{S_{n+1}}|\mathcal{F}_n) = 2^{S_n}E2^{X_{n+1}} = 2^{S_n}(2^{-1}p + 2^0(1 - 2p) + 2p) = Y_n(1 + \tfrac{p}{2})$$

we see that we have a submatingale for any given $p \in (0, \frac{1}{2}]$.

5b. By Doob's decomposition,

$$Y_n = M_n + C_n,$$

where M_n is a martingale, the compensator for (Y_n) is computed recursively from

$$C_0 = 0, \quad C_{n+1} - C_n = E(Y_{n+1}|\mathcal{F}_n) - Y_n = Y_n\tfrac{p}{2},$$

so that

$$C_n = \tfrac{p}{2}(Y_0 + \ldots + Y_{n-1}) = \tfrac{p}{2}(1 + \sum_{k=1}^{n-1} 2^{X_1+\ldots+X_k}).$$

6c. Eventhough the mean $EW_n = 0$ for all n, the variance can not be constant since

$$\mathrm{Var}W_n = \mathrm{Var}W_0 + 1 > \mathrm{Var}W_0.$$

So the sequence $(W_n)_{n\geq 0}$ can not be stationary.