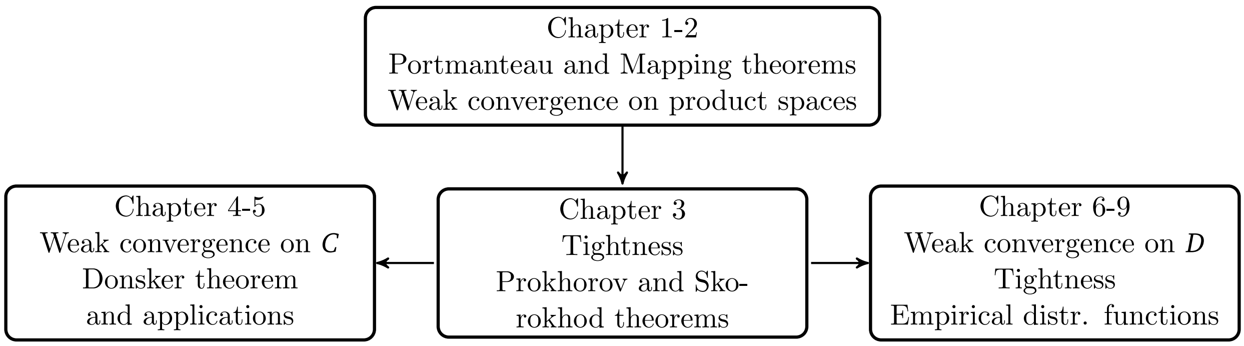
Digest of the lecture notes for the graduate course Weak Convergence

Olle Elias

We first outline the logical progression of the lecture notes and highlight the most important parts in each chapter. Then we give a more detailed overview for each part seen in the next diagram.



The three first chapters is the theoretical backbone of the remaining chapters. The topological framework in these notes will be restricted to complete separable metrics spaces, also known as Polish spaces, but the definition of weak convergence do not require any notion of metric.

The first chapter contains the Portmanteau's theorem, **Theorem 1.15**, and the Mapping theorem, **Theorem 1.18**, which are perhaps the two most fundamental results in these notes.

Chapter 2 is titled *Convergence of finite-dimensional distributions* but the most important part of this chapter is perhaps is the concepts of separating and convergence-determining classes, and weak convergence on product spaces.

The third chapter might be the most important one out of the three and contains the concept of tightness and Prokhorov's theorem, **Theorem 3.9**, both of which plays a central role in the remaining chapters. One must also mention Skorokhod's theorem, **Theorem 3.11**, even though its implications are not covered in these notes.

The remaining chapters can essentially be divided into two independent parts, weak convergence on the space of continuous functions and the space of càdlàg functions. The ideas and arguments are rather similar but the theory of càdlàg functions are much more technical involved so it is suggested to begin with the theory of continuous functions.

Chapter 4 deal with the space of continuous functions and the key results are Donskers theorem/Functional CLT, **Theorem 4.17**, and the Arzela-Ascoli theorem, **Theorem 4.23**, which is vital to characterizing tightness.

Chapter 5 deals with applications of the functional CLT such as constructing the Brownian bridge measure, **Theorem 5.10**. Chapter 6 through 8 deals with the space of càdlàg functions on the unit interval, Chapter 6 discusses different metrics and relatively compactness, and Chapter 7 characterizes tightness, **Theorem 7.11**. Chapter 8 consists of general weak convergence results, the functional CLT for càdlàg functions, **Theorem 8.4**, and a convergence result of the empirical distribution function, **Theorem 8.11**. The final chapter extends the general results given in chapter 6-8 on the positive real axis.

Chapter 1-2

The first chapter is divided into three sections: Metric spaces, Weak convergence, and convergence in probability and total variation. The first section consists of various results and definitions on

Polish spaces as well as the definition of π- and λ-systems and Dynkins $\pi - \lambda$ lemma. Theorem 1.7 is of special importance since this result is crucial for the proof of Prokhorovs theorem in Chapter 3. The second section of this chapter contains three important results, namely Portmanteau's theorem, regularity of probability measures (Corollary 1.17) and the Mapping theorem. The last section of Chapter 1 present other types of convergence as well as a local central limit theorem on lattice's, Theorem 1.27, which finds its use in Chapter 5.

The second chapter contains four sections: Separating and convergence-determining classes, Weak convergence in product spaces, Weak convergence in $\mathbb{R}^k$ and $\mathbb{R}^\infty$, and Kolmogorov's extension theorem. The first section deals with two different type of subclasses of the Borel σ-algebra, namely the separating class and the convergence-determining class, both of which are important in when it comes to weak convergence on C and D. The second and third section deals with weak convergence on product spaces, Theorem 2.7 gives a characterization of convergence of product measures whereas the third part deals with $\mathbb{R}^k$ and $\mathbb{R}^\infty$ more concretely but some of the ideas can be transferred to the last part of these notes when dealing with the space D_∞. The last section of Chapter 2 contains the statement of Kolmogorov's extension theorem.

Chapter 3

Chapter 3 essentially consists of the definition of tightness, the Prokhorov distance and some results regarding these, the proof of Prokhorov's theorem, and Skorokhod's theorem. As mentioned, the first section defines tightness as well as the Prokhorov distance and states a result in fact shows that weak convergence is equivalent convergence in the Prokhorov distance, it additionally shows the converse of Prokhorov's theorem, namely any family of probability measures that is relatively compact is tight.

The second sections contains the proof of Prokhorov's theorem which follows the following argument: Let Π be a tight family of probability measures and let P_n be a sequence in Π. The idea is to construct a limiting object $\alpha(\cdot)$ from the sequence P_n on a certain family of sets, $\mathcal{H} = (H_i)_i$ using Theorem 1.7, and then show that this object can be used to define probability measure through the two set functions $\beta(G) = \sup_{H \subset G} \alpha(H)$ and $\gamma(M) = \inf_{G \supset M} \beta(G)$. After constructing α, β, γ the proof is divided into the following seven steps.

Step 1 If F is closed and G is open such that $F \subset H$ for some $H \in \mathcal{H}$ then there exists a $H_0 \in \mathcal{H}$ such that $F \subset H_0 \subset G$.

Step 2 β is finitely subadditive on open sets.

Step 3 β is countably subadditive on open sets.

Step 4 γ is an outer measure.

Step 5 $\beta(G) \geq \gamma(F \cap G) + \gamma(F^c \cap G)$ for F closed and G open.

Step 6 If F is closed , then F is γ-measurable.

Step 7 The restriction P of γ is a probability measure satisfying $P(G) = \gamma(G) = \beta(G)$ for all open sets G.

The last sections of this chapter contains the Skorokhod's representation theorem which shows that given a weakly convergent sequence of probability measures $(P_n)_n$, $P = \lim_n P_n$, one can construct a sequence of random elements $(X_n)_n, X = \lim_n X_n$ that converge almost surely such that the law X_n is P_n and the law of X is P. Similarly to Prokhorovs one can divide the proof into steps:

Step 1 Show that, for each $\epsilon > 0$ there is a finite Borel-measurable partition, $(B_i)_1^k$ such that

$$P(B_0) < \epsilon,\ P(\partial B_i) = 0,\ \operatorname{diam}(B_i) < \epsilon,\ i = 1,2,\ldots,k.$$

Step 2 Define the sequence (n_j).

Step 3 Construction of X, Y_n, Y_{ni}, Z_n, ξ.

Step 4 Constructing the sequence (X_n).

Chapter 4-5

The remaining parts of the notes are rather similar and can be read independently of each other, though it is probably wise to begin here since it is less technical and the ideas are essentially the same but clearer. Chapter 4 is divided into three sections with the main focus of proving the functional central limit theorem on C, also known as Donskers theorem or the invariance principle.

The first section has three major results: C is a Polish space with the uniform norm (it is in fact a Banach space), a criterion for weak convergence, Theorem 4.12, and that the sets $\pi^{-1}_{t_1,\dots,t_k}(H)$ form a separating class.

The second section contains the statement of Donker's theorem and a proof based on Theorem 4.12 using a maximal inequality for random walks known as Etemadi's inequality.

The last section contains two important results, The Arzela-Ascoli theorem on C which characterizes sets that are relatively compact and a theorem that generalizes Theorem 4.12, and which gives a characterization of tight families of probability measures on C using Arzela-Ascoli's theorem.

Chapter 5 contains three major results that follow from applying the invariance principle. The first is the minimum and maximum of the Brownian path which is obtained through deriving a similar result for the symmetric simple random walk and applying the mapping theorem. The second result is the arcsine law which derives the joint density of the random times:

$$h_1(x) = \sup\{t : x(t) = 0, t \in [0,1]\},$$
$$h_2(x) = |\{t : x(t) > 0, t \in [0,1]\}|,$$
$$h_3(x) = |\{t : x(t) > 0, t \in [0,h_1(x)]\}|.$$

The derivation is similar to the previous result but here one have to apply the local limit theorem, Theorem 1.27. The final result is proving the existence of the Brownian bridge measure which essentially only relies on Portmanteau's theorem.

Chapter 6-9

The final part of these notes deal with the space of càdlàg functions, that is functions that are right continuous and have limits from the left. The first three chapters deal with the càdlàg functions on the unit interval, whereas the final chapter extends the results obtained in chapters $6-8$ to càdlàg functions on $[0,\infty)$. Essentially all major results and proofs are analogous to the corresponding statement on C.

Chapter 6 establishes the fundamental properties of the space D. The key points of this chapter is the relation between the functionals $w_x(\delta), w'_x(\delta), w''_x(\delta), x \in D, \delta \in (0,1)$, the metrics d, d^o and D as a metric space under d and d^o respectively, and an analogue of Arzela-Ascoli on D.

Chapter 7 deals with probability measures on D and in the first section establishes analogous results to Lemma 4.13 and Theorem 4.15 which states that projections are a convergence-determining class and a separating class. The second section of Chapter 7 characterizes tightness in D and the statement, as well as the proof is essentially analogous to the statement in C. The last two sections of Chapter 7 deals with existence of random elements given a consistent family of finite dimensional distributions.

The main point of Chapter 8 is to establish the functional CLT on D and due to the more general setting this allows us to prove the CLT in greater generality, see Theorem 8.7, which shows a much wider class of random walks converges to the Wiener process. The chapter ends with proving a CLT for empirical distribution functions, Theorem 8.11, which establishes a connection with the Brownian Bridge and the Kolmogorov-Smirnoff test.

The final chapter, as mentioned, extends the results to the positive real axis. The idea is to consider $D[0,\infty)$ as closed set in the product space $\prod_1^\infty D_i$ where $D_i = D[0,i]$ and then apply the theory developed in Chapter 2.