Lösningar MSG110

1.) Låt T = antalet vandrare under 100 dagar.

T är $Po(100 \cdot 1) = Po(100)$. Eftersom $100 > 15$ är

$T \approx N(100, 100)$. Vi får $P(T \leq 110) =$

$P\left(\frac{T-100}{10} \leq \frac{110-100}{10}\right) \approx \Phi(1) \underset{\text{tabell}}{=} \boxed{0,8413}$

2.) a) Likelihoodfunktionen $L(\alpha) = f(x_1) f(x_2) =$

$= (\alpha-1) 2^{-\alpha} (\alpha-1) 4^{-\alpha} = (\alpha-1)^2 8^{-\alpha}$.

$\ell(\alpha) = \ln L(\alpha) = 2\ln(\alpha-1) - \alpha \ln 8$

$\ell'(\alpha) = \frac{2}{\alpha-1} - \ln 8 = 0 \Leftrightarrow \alpha \ln 8 - \ln 8 = 2 \Leftrightarrow$

$\Leftrightarrow \alpha = \frac{2+\ln 8}{\ln 8} = \boxed{\frac{2}{3\ln 2} + 1} \approx 1,962$

b) $E(X) = \int_1^\infty x (\alpha-1) x^{-\alpha} dx = (\alpha-1) \left[\frac{x^{-\alpha+2}}{2-\alpha}\right]_1^\infty$

$\underset{\alpha > 2}{=} \frac{\alpha-1}{\alpha-2}$. $\quad \bar{x} = \frac{2+4}{2} = 3$.

$\frac{\alpha-1}{\alpha-2} = 3 \Leftrightarrow \alpha - 1 = 3\alpha - 6 \Leftrightarrow 2\alpha = 5 \Leftrightarrow \alpha = \boxed{2,5}$

3.) a) Ett 95% ki är $\bar{x} \pm t_{0,025}(3) \frac{s}{\sqrt{n}} = 33 \pm 3,1824 \cdot \frac{2}{2}$

$= \boxed{33 \pm 3,1824}$

b) Beräkna först ett 90% nedåt begränsat k.i.:

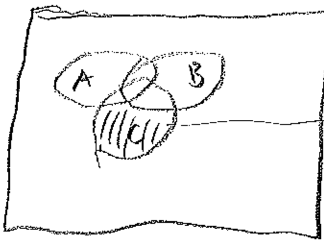
$[\bar{x} - t_{0,1}(3) \cdot \frac{s}{\sqrt{n}}, \infty) = [33 - 1,6377, \infty) = [31,3623, \infty)$

Eftersom $30 < 31,3623$ kan H_0 förkastas till förmån för H_1 på nivå $\alpha = 0,1$.

4. a) Additionssatsen $\Rightarrow$ $P(A\cup B) = P(A)+P(B) - P(A\cap B)$

$$\underset{\substack{\uparrow \\ \text{oberoende}}}{=} 0{,}3+0{,}3-0{,}3^2 = 0{,}6-0{,}09 = \boxed{0{,}51}$$

b)



$A^c \cap B^c \cap C$

Venndiagrammet ger: $P(A^c\cap B^c \cap C) = P(A\cup B\cup C) - P(A\cup B)$
$= 0{,}71 - 0{,}51 = \boxed{0{,}2}$

5. a) $$\int_0^1\int_0^1 \frac{4}{5}(xy+1)\,dx\,dy = \frac{4}{5}\int_0^1 \left[\frac{x^2y}{2}+x\right]_0^1 dy =$$

$$= \frac{4}{5}\int_0^1 \frac{1}{2}y+1\,dy = \frac{4}{5}\left[\frac{y^2}{4}+y\right]_0^1 = \frac{4}{5}\left(\frac{1}{4}+1\right) = \frac{4}{5}\cdot\frac{5}{4} = 1.$$

Dessutom är $f(x,y)\geq 0$ trivialt för $x,y\in[0,1]$.

b) $C(X,Y) = E(XY) - E(X)E(Y)$

$$E(XY) = \int_0^1\int_0^1 xy\,\frac{4}{5}(xy+1)\,dx\,dy = \frac{4}{5}\int_0^1\int_0^1 x^2y^2+xy\,dx\,dy$$

$$= \frac{4}{5}\int_0^1\left[\frac{x^3y^2}{3}+\frac{x^2y}{2}\right]_0^1 dy = \frac{4}{5}\int_0^1 y^2/3 + y/2\,dy = \frac{4}{5}\left[\frac{y^3}{9}+\frac{y^2}{4}\right]_0^1$$

$$= \frac{4}{5}\left(\frac{1}{9}+\frac{1}{4}\right) = \frac{4}{5}\left(\frac{4}{36}+\frac{9}{36}\right) = \frac{4\cdot 13}{5\cdot 36} = \frac{13}{45}$$

$$E(X) = \int_0^1\int_0^1 x\,\frac{4}{5}(xy+1)\,dx\,dy = \frac{4}{5}\int_0^1\int_0^1 x^2y+x\,dx\,dy =$$

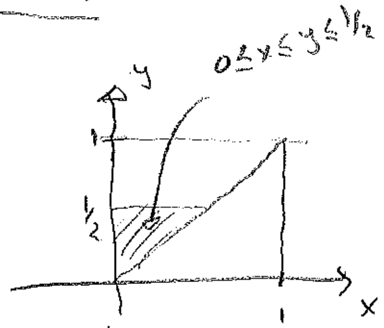
$$= \frac{4}{5}\int_0^1\left[\frac{x^3y}{3}+\frac{x^2}{2}\right]_0^1 dy = \frac{4}{5}\int_0^1 \frac{y}{3}+\frac{1}{2}\,dy = \frac{4}{5}\left[\frac{y^2}{6}+\frac{y}{2}\right]_0^1$$

$= \frac{4}{5}\left(\frac{1}{6}+\frac{1}{2}\right) = \frac{4}{5}\cdot\frac{4}{6} = \frac{8}{15}$. Samma slags uträkning ger att

$E(Y) = \frac{8}{15}$. $\Longrightarrow$ $C(X,Y) = \frac{13}{45} - \left(\frac{8}{15}\right)^2 = \frac{13}{45} - \frac{64}{225} = \frac{65-64}{225} = \boxed{\frac{1}{225}}$

5 c) $P(X \le Y \mid Y \le 1/2) = \dfrac{P(\{X \le Y\} \cap \{Y \le 1/2\})}{P(Y \le 1/2)}$

$$= \frac{P(X \le Y \le 1/2)}{P(Y \le 1/2)} = A$$



$$P(X \le Y \le 1/2) = \int_{y=0}^{1/2} \int_0^y \frac{4}{5}(xy+1)\,dx\,dy = \frac{4}{5}\int_0^{1/2} \left[\frac{x^2 y}{2} + x\right]_0^y dy$$

$$= \frac{4}{5}\int_0^{1/2} \frac{y^3}{2} + y \, dy = \frac{4}{5}\left[\frac{y^4}{8} + \frac{y^2}{2}\right]_0^{1/2} = \frac{4}{5}\left(\frac{1}{16\cdot 8} + \frac{1}{8}\right)$$

$$= \frac{1}{5}\left(\frac{1}{32} + \frac{1}{2}\right) = \frac{17}{5\cdot 32} = \frac{17}{160}$$

$$P(Y \le 1/2) = \int_0^{1/2}\int_0^1 \frac{4}{5}(xy+1)\,dx\,dy = \frac{4}{5}\int_0^{1/2} \left[\frac{x^2 y}{2} + x\right]_0^1 dy =$$

$$= \frac{4}{5}\int_0^{1/2} \frac{y}{2} + 1 \, dy = \frac{4}{5}\left[\frac{y^2}{4} + y\right]_0^{1/2} = \frac{4}{5}\left(\frac{1}{16} + \frac{1}{2}\right) = \frac{4}{5}\cdot\frac{9}{16} = \frac{9}{20}$$

$$\Longrightarrow A = \frac{(17/160)}{(9/20)} = \boxed{\frac{17}{72}} \approx 0{,}236$$

6. Låt X = antal gröna bollar som dras från urna 1
Y = — " — 2

a) P(samtliga som dras från urnan 2) = $P(Y=2)$

$$\underset{\substack{\uparrow\\ \text{Lagen om total sannolikhet}}}{=} P(Y=2 \mid X=0)P(X=0) + P(Y=2 \mid X=1)P(X=1)$$

$$= \left(\frac{4}{7}\cdot\frac{3}{6}\right)\cdot\left(\frac{4}{5}\cdot\frac{3}{4}\right) + \frac{5}{7}\cdot\frac{4}{6}\left(1-\frac{4\cdot 3}{5\cdot 4}\right)$$

$$= \frac{36}{210} + \frac{10}{21}\cdot\frac{2}{5} = \frac{36}{210} + \frac{20}{105} = \frac{36+40}{210} = \frac{76}{210} = \boxed{\frac{38}{105}}$$

b) Vi söker nu $P(X=1 \mid Y=2)$.

Bayes sats $\Rightarrow$ $P(X=1 \mid Y=2) = \dfrac{P(Y=2 \mid X=1)P(X=1)}{P(Y=2 \mid X=1)P(X=1) + P(Y=2 \mid X=0)P(X=0)}$

$$\underset{\substack{\uparrow\\ a)}}{=} \frac{20/105}{38/105} = \frac{20}{38} = \boxed{\frac{10}{19}}$$

7. Låt X = antal kast Anna behöver
Y = — " — Anders.

Då är $X \sim ffg(0{,}5)$ och $Y \sim ffg(0{,}4)$

a) $P(X \le 3) = P(X=1) + P(X=2) + P(X=3) = 0{,}5 + 0{,}5^2 + 0{,}5^3 =$
$= 0{,}5 + 0{,}25 + 0{,}125 = \boxed{0{,}875}$

b) $P(\text{oavgjort}) = P(X=Y) = P\left(\bigcup_{k=1}^{\infty}\left(\{X=k\} \cap \{Y=k\}\right)\right)$

$$= \sum_{k=1}^{\infty} P(\{X=k\} \cap \{Y=k\}) \underset{\substack{\uparrow\\ \text{ober.}}}{=} \sum_{k=1}^{\infty} P(X=k)P(Y=k) =$$

$$= \sum_{k=1}^{\infty} 0{,}5\cdot 0{,}5^{k-1}\cdot 0{,}4\cdot 0{,}6^{k-1} = \frac{0{,}4}{0{,}6}\sum_{k=1}^{\infty}(0{,}5\cdot 0{,}6)^k =$$

$$= \frac{2}{3}\cdot\sum_{k=1}^{\infty} 0{,}3^k = \frac{2}{3}\cdot\frac{0{,}3}{0{,}7} = \frac{2}{3}\cdot\frac{3}{7} = \boxed{\frac{2}{7}}$$

8.) Antag X kontinuerlig med sannolikhetsfunktion $f(x)$.

Vi vet att $g(x) \geq g(y) + g'(y)(x-y)$ för $x, y \in \mathbb{R}$

Vi får att $E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx \geq \int_{-\infty}^{\infty} (g(y) + g'(y)(x-y)) f(x)\,dx$

$$= \int_{-\infty}^{\infty} g(y) f(x)\,dx + \int_{-\infty}^{\infty} g'(y) x f(x)\,dx - \int_{-\infty}^{\infty} g'(y) y f(x)\,dx$$

$$= g(y) \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1} + g'(y) \underbrace{\int_{-\infty}^{\infty} x f(x)\,dx}_{=E(X)} - g'(y) y \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1}$$

$= g(y) + g'(y)(E(X) - y)$. Dvs, för alla $y \in \mathbb{R}$ gäller $E(g(X)) \geq g(y) + g'(y)(E(X) - y)$.

Om vi låter $y = E(X)$ så fås att

$$E(g(X)) \geq g(E(X)).$$

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