THEORETICAL QUESTIONS FOR THE WRITTEN EXAMINATION IN BASIC STOCHASTIC PROCESSES 2009

Question 1.

(a) Give two different definition of the Poisson process $\{N(t),\, t \geq 0\}$ with rate λ.

(b) State and prove the memoryless property of the process (Theorem 1.1.2).

Question 2. $\{N(t),\, t \geq 0\}$ is the Poisson process of rate λ. Let $S_1,\, S_2, \ldots$ be the arrival times of the process and set $S_0 = 0$.

(a) State and prove the result about the conditional distribution

$$P(S_k \leq x | N(t) = n)$$

and the conditional expectations

$$E[S_k - S_{k-1} | N(t) = n], \quad 1 \leq k \leq n$$

(Lemma 1.1.4).

(b) State without proof the result about the conditional joint distribution

$$P(S_1 \leq x_1, \ldots S_n \leq x_n | N(t) = n)$$

(Theorem 1.1.5).

Question 3.

(a) Give a definition of the compound Poisson process $\{X(t),\, t \geq 0\}$. (Definition 1.2.1). Give a formula for $E[X(t)]$ (formula (1.2.1)) and prove it.

(b) Suppose the jumps $D_1,\, D_2,\, \ldots$ of the compound Poisson process are integer-valued with

$$a_j = P\{D_1 = j\}, \;\; j = 0,\, 1,\, \ldots$$

and for any $t \geq 0$, let $\quad r_j(t) = P\{X(t) = j\}, \;\; j = 0,\, 1,\, \ldots$

Prove that the generating functions

$$A(z) = \sum_{j=0}^{\infty} a_j z^j \text{ and } R(z,t) = \sum_{j=0}^{\infty} r_j(t) z^j, \quad \text{where} \quad |z| \leq 1,$$

satisfy $\quad R(z,t) = e^{-\lambda t[1-A(z)]}, \;$ for any $t > 0$ (Theorem 1.2.1 (a)).

Question 4.

(a) Give a definition of a renewal process $\{N(t),\, t \geq 0\}$ and its renewal function $M(t)$.

(b) For $n = 1,\, 2,\, \ldots$ let $F_n(t)$ be the distribution function of the renewal time S_n. Give a formula relating these functions and the function $M(t)$ and prove it (Lemma 2.1.1).

(c) Let μ be the average interocurrence time of the process. Fix $t > 0$ and consider the excess variable $\gamma_t = S_{N(t)+1} - t$. Show that (Lemma 2.1.2)

$$E[\gamma_t] = \mu[1 + M(t)] - t$$

Question 5.

(a) Give a definition of a regenerative stochastic process $\{X(t),\, t \geq 0\}$.

(b) Suppose $C_1,\, C_2,\, \ldots$ are the lengths of the renewal cycles of the regenerative process and assume that $E[C_1] < \infty$. Prove the following (Lemma 2.2.2):

For any $t > 0$, the number $N(t)$ of cycles completed up to the moment t satisfies

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{E[C_1]}$$

(c) State without proof the renewal-reward theorem (Theorem 2.2.1).

Question 6. Define the homogeneous discrete-time Markov chain $\{X_n,\, n = 0,\, 1,\, \ldots\}$ and its n-step transition probabilities $p_{ij}^{(n)}$. State and prove the Chapman-Kolmogoroff equations for the transition probabilities of the process (Theorem 3.2.1).

Question 7. Prove the following result.

Lemma A. Assume the state r of a discrete-time Markov chain $\{X_n,\, n = 0,\, 1,\, \ldots\}$ with a finite state space I is accessible from each state in I. Consider the first-visit time to r

$$\tau = \min\{n \geq 1 : X_n = r\}$$

and the mean first-visit times from i to r

$$\mu_{ir} = E[\tau | X_0 = i], \quad i \in I.$$

It holds

$$\mu_{rr} = 1 + \sum_{j \in I,\, j \neq r} p_{rj} \mu_{jr}.$$

Proof. We have

$$\mu_{rr} = E[\tau | X_0 = r] = \sum_{k=1}^{\infty} k P\{\tau = k \mid X_0 = r\} = p_{rr} + \sum_{k=2}^{\infty} k P\{\tau = k \mid X_0 = r\}. \qquad (1)$$

We compute $P\{\tau = k|X_0 = r\}$ for $k \geq 2$. Since the first visit to r is not on the first step, we have the representation

$$\{\tau = k\} = \cup_{j \in I,\, j \neq r} \{\tau = k,\ X_1 = j\}.$$

where the events in the right-hand side are disjoint. Thus

$$\begin{aligned} P\{\tau = k \mid X_0 = r\} &= \sum_{j \in I,\, j \neq r} P\{\tau = k,\ X_1 = j \mid X_0 = r\} \\ &= \sum_{j \in I,\, j \neq r} \frac{P\{\tau = k,\ X_1 = j,\ X_0 = r\}}{P\{X_0 = r\}} \\ &= \sum_{j \in I,\, j \neq r} P\{\tau = k|X_1 = j,\ X_0 = r\} \frac{P\{X_1 = j,\ X_0 = r\}}{P\{X_0 = r\}} \\ &= (\text{markov property}) = \sum_{j \in I,\, j \neq r} P\{\tau = k|X_1 = j\}\, p_{rj} \\ &= \sum_{j \in I,\, j \neq r} P\{\text{first visit to } r \text{ in } k-1 \text{ steps}|X_1 = j\}\, p_{rj} \\ &= \sum_{j \in I,\, j \neq r} P\{\tau = k-1|X_0 = j\}\, p_{rj}. \end{aligned} \tag{2}$$

Next we replace in (1) the terms $P\{\tau = k \mid X_0 = r\}$ by the expression obtained in (2).

$$\begin{aligned} \mu_{rr} &= p_{rr} + \sum_{k=2}^{\infty} k \sum_{j \in I,\, j \neq r} P\{\tau = k-1|X_0 = j\}\, p_{rj} \\ &= p_{rr} + \sum_{j \in I,\, j \neq r} p_{rj} \sum_{k-1=1}^{\infty} [(k-1)+1] P\{\tau = k-1|X_0 = j\} \\ &= (\text{substitution } k-1 = m) = p_{rr} + \sum_{j \in I,\, j \neq r} p_{rj} \left[\sum_{m=1}^{\infty} m P\{\tau = m|X_0 = j\} + \sum_{m=1}^{\infty} P\{\tau = m|X_0 = j\} \right] \\ &= p_{rr} + \sum_{j \in I,\, j \neq r} p_{rj} \Big[E[\tau|X_0 = j] + 1 \Big] \\ &= p_{rr} + \sum_{j \in I,\, j \neq r} p_{rj}\mu_{jr} + \sum_{j \in I,\, j \neq r} p_{rj} = 1 + \sum_{j \in I,\, j \neq r} p_{rj}\mu_{jr}. \end{aligned}$$

□

Question 8. Prove the following result.

Theorem 3.3.1, first part. Let $\{X_n,\ n = 0,\ 1,\ \dots\}$ be a discrete-time Markov chain and j be any state of the process. The k-step transition probabilities $p_{jj}^{(k)},\ \ k = 1,\ 2,\ \dots$ and the mean return time μ_{jj} of j satisfy

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} p_{jj}^{(k)} = \begin{cases} \frac{1}{\mu_{jj}}, & \text{if state j is recurrent} \\ 0, & \text{if state j is transient} \end{cases}$$

Question 9. Analysis of the M/M/1 queuing system by using a Markov chain model.

(a) Describe the M/M/1 queuing system. For any $t \geq 0$, let $X(t) =$ the number of customers present at time t. Derive the infinitesimal transition rates of the process and sketch the state diagram.

(b) Explain, under what assumption has the process equilibrium probabilities and compute these probabilities.

(c) Explain the formula that can be used to compute the long-rum average number of customers in queue and compute this number. What is the long-run fraction of customers who find j other customers present upon arrival?