Solution to the written examination in MVE170/MSG800
BASIC STOCHASTIC PROCESSES
December 14, 2009

Problem 1.

(a) Give two different definitions of the Poisson process $\{N(t),\, t \geq 0\}$ with rate λ.

(b) State without proof the memoryless property of the process.

(c) Explain the PASTA property [Poisson arrivals see time averages] of the Poisson process and comment on it. 3p

Problem 2.

(a) Define the homogeneous discrete-time Markov chain $\{X_n,\, n = 0,\, 1,\, \ldots\}$ and its n-step transition probabilities $p_{ij}^{(n)}$.

(b) State and prove the Chapman-Kolmogoroff equations for the transition probabilities of the process. 5p

For answers to problems 1-2 see the course book.

Problem 3. An information centre has one attendant; people with questions arrive according to a Poisson process with rate λ. A person who finds n other customers present upon arrival joins the queue with probability $1/(n+1)$ for $n = 0,\, 1,\, \ldots$ and goes elsewhere otherwise. The service times of the persons are independent random variables having an exponential distribution with mean $1/\mu$.

(a) Formulate a continuous-time Markov chain to analyse the number of persons present at the information centre and specify the transition rate diagram.

(b) Find the equilibrium distribution of the process.

(c) What is the long-run fraction of persons with request who actually join the queue? Explain your answer.

(d) What is the long-run average number of persons served per time unit? Explain.

7p

Solution

(a) Let $X(t)$ be the number of persons present at time t. The process $\{X(t),\, t \geq 0\}$ is a continuous-time Markov chain with state space $I = \{0,\, 1,\, 2,\, \ldots\}$. The transition rate diagram is given by

$$\boxed{0} \;\; \underset{\mu}{\overset{\lambda/1}{\rightleftarrows}} \;\; \boxed{1} \quad \ldots \quad \boxed{(i-1)} \;\; \underset{\mu}{\overset{\lambda/i}{\rightleftarrows}} \;\; \boxed{i} \quad \ldots$$

(b) By equating the rate out of the set $\{i,\, i+1,\, \ldots\}$ to the rate into this set, we find the recurrence relations

$$\mu p_i = \frac{\lambda}{i}\, p_{i-1}, \;\; i = 1,\, 2,\, \ldots$$

These equations lead to

$$p_i = \frac{(\lambda/\mu)^i}{i!} p_0, \quad i \geq 1.$$

Using the normalizing equation $\sum p_i = 1$ we obtain

$$p_i = e^{-\lambda/\mu} \frac{(\lambda/\mu)^i}{i!}, \quad i \geq 0.$$

(c) By the PASTA property, the long-run fraction of arrivals that actually join the queue is

$$\sum_{i=0}^{\infty} p_i \, \frac{1}{i+1} = \frac{\mu}{\lambda}\big(1 - e^{-\lambda/\mu}\big).$$

(d) The long-run average number of persons served per time unit is

$$\lambda \left[\frac{\mu}{\lambda}\big(1 - e^{-\lambda/\mu}\big)\right] = \mu(1 - p_0),$$

in agreement with the Little's formula.