

Solution to the written examination in MVE170/MSG800

BASIC STOCHASTIC PROCESSES

December 14, 2009

Problem 1.

- (a) Give two different definitions of the Poisson process $\{N(t), t \geq 0\}$ with rate λ .
- (b) State without proof the memoryless property of the process.
- (c) Explain the PASTA property [Poisson arrivals see time averages] of the Poisson process and comment on it. 3p

Problem 2.

- (a) Define the homogeneous discrete-time Markov chain $\{X_n, n = 0, 1, \dots\}$ and its n -step transition probabilities $p_{ij}^{(n)}$.
- (b) State and prove the Chapman-Kolmogoroff equations for the transition probabilities of the process. 5p

For answers to problems 1-2 see the course book.

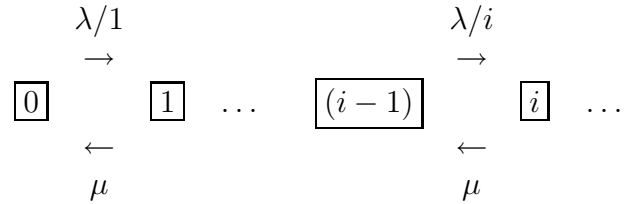
Problem 3. An information centre has one attendant; people with questions arrive according to a Poisson process with rate λ . A person who finds n other customers present upon arrival joins the queue with probability $1/(n+1)$ for $n = 0, 1, \dots$ and goes elsewhere otherwise. The service times of the persons are independent random variables having an exponential distribution with mean $1/\mu$.

- (a) Formulate a continuous-time Markov chain to analyse the number of persons present at the information centre and specify the transition rate diagram.
- (b) Find the equilibrium distribution of the process.
- (c) What is the long-run fraction of persons with request who actually join the queue? Explain your answer.
- (d) What is the long-run average number of persons served per time unit? Explain.

7p

Solution

- (a) Let $X(t)$ be the number of persons present at time t . The process $\{X(t), t \geq 0\}$ is a continuous-time Markov chain with state space $I = \{0, 1, 2, \dots\}$. The transition rate diagram is given by



- (b) By equating the rate out of the set $\{i, i + 1, \dots\}$ to the rate into this set, we find the recurrence relations

$$\mu p_i = \frac{\lambda}{i} p_{i-1}, \quad i = 1, 2, \dots$$

These equations lead to

$$p_i = \frac{(\lambda/\mu)^i}{i!} p_0, \quad i \geq 1.$$

Using the normalizing equation $\sum p_i = 1$ we obtain

$$p_i = e^{-\lambda/\mu} \frac{(\lambda/\mu)^i}{i!}, \quad i \geq 0.$$

- (c) By the PASTA property, the long-run fraction of arrivals that actually join the queue is

$$\sum_{i=0}^{\infty} p_i \frac{1}{i+1} = \frac{\mu}{\lambda} (1 - e^{-\lambda/\mu}).$$

- (d) The long-run average number of persons served per time unit is

$$\lambda \left[\frac{\mu}{\lambda} (1 - e^{-\lambda/\mu}) \right] = \mu(1 - p_0),$$

in agreement with the Little's formula.