Basic stochastic processes

Solution to take home re-examination in April 2010

Day assigned: April 8, 10:00. **Due date:** April 9, 10:00 am

• The take home examination is a strictly individual assignment. Submissions that bear signs of being collective efforts will be disregarded

• Answers without explanations will be disregarded as well.

Problem 1. A Bernoulli trial results in a success with probability p and in a failure with probability $1-p$, where $0 < p < 1$. Suppose the Bernoulli trial is repeated indefinitely with each repetition independent of all others. Let X_n be a "success runs" Markov chain with a state space $I = \{0, 1, 2, \dots\}$, where $X_n = 0$ if the $n-th$ trial results in a failure and $X_n = j$ if $X_{n-j} = 0$ and trials $n-j+1, \dots, n$ have resulted in a success. Find the one-step transition matrix of the Markov chain. Show that all states are recurrent. 5p

Solution

For $i,\, j \in I$ the one-step transition probabilities are

$$p_{ij} = \begin{cases} p & \text{if} \quad j = i+1 \\ 1-p & \text{if} \quad j = 0 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$\begin{aligned} f_{00}^{(n)} &= P\{X_n = 0,\, X_{n-k} \neq 0,\, 1 \le k \le (n-1) | X_0 = 0\} \\ &= P\{(n-1) \ \text{ successes followed by 1 failure } \} = p^{n-1}(1-p) \end{aligned}$$

and

$$\sum_{n=1}^{\infty} f_{00}^{(n)} = \sum_{n=1}^{\infty} p^{n-1}(1-p) = 1,$$

thus state 0 is recurrent. Since all states communicate with one another they are recurrent.

Problem 2. In the beginning of each time unit a job arrives at a conveyor with a single work station. The workstation can process only one job at a time and has a buffer for waiting jobs, that can hold at most K jobs. Any arriving gob that finds the buffer full is lost. The processing times are independent and have exponential distribution with mean $1/\mu$. It is assumed that $\mu > 1$. Define a Markov chain to analyse the number of jobs in the buffer just prior to the arrival epochs of new jobs and specify the one-step transition probabilities. Show how to calculate the long-run fraction of lost jobs and the long-run fraction of time the workstation is busy. 5p

Solution

Let X_n=the # of jobs in the system just prior to the $n-th$ arrival. $\{X_n\}$ is a Markov chain with state space $I = \{0, 1, \ldots K+1\}$. The one-step transition probabilities are as follows.
For $0 \le i \le K$

$$p_{ij} = e^{-\mu}\frac{\mu^{i+1-j}}{(i+1-j)!}, \quad 1 \le j \le i+1,$$
$$p_{i0} = 1 - \sum_{j=1}^{i+1} e^{-\mu}\frac{\mu^{i+1-j}}{(i+1-j)!}.$$

For $i = K+1$

$$p_{K+1,j} = e^{-\mu}\frac{\mu^{K+1-j}}{(K+1-j)!}, \quad 1 \le j \le K+1,$$
$$p_{K+1,0} = 1 - \sum_{j=1}^{K+1} e^{-\mu}\frac{\mu^{i+1-j}}{(i+1-j)!}.$$

Since $\mu > 1 \ge$ the rate of accepted messages, the system has equilibrium probabilities $\{\pi_i,\ 0 \le i \le K+1\}$. By the PASTA property

$$\text{the long-run fraction of jobs rejected} = \pi_{K+1}.$$

The rate of accepted jobs is then $1 - \pi_{K+1}$. By the Little's formula

$$\text{the long-run fraction of time the station is busy} = \frac{1}{\mu}[1 - \pi_{K+1}].$$

Problem 3. An information centre has one attendant; people with questions arrive according to a Poisson process with rate λ. A person who finds n other customers present upon arrival joins the queue with probability $1/(n+1)$ for $n = 0, 1, \ldots$ and goes elsewhere otherwise. The service times of the persons are independent random variables having an exponential distribution with mean $1/\mu$. Verify that the equilibrium distribution of the number of persons present at the information centre is a Poisson distribution with mean λ/μ. What is the long-run fraction of persons with request who actually join the queue? What is the long-run average number of persons served per time unit? Explain your answers. 5p

Solution Let $X(t)$ be the number of persons present at time t. The process $\{X(t),\, t \ge 0\}$ is a continuous-time Markov chain with state space $I = \{0, 1, 2, \ldots\}$. The transition rate diagram is given by

$$\boxed{0} \underset{\mu}{\overset{\lambda/1}{\rightleftarrows}} \boxed{1} \ \cdots \ \boxed{(i-1)} \underset{\mu}{\overset{\lambda/i}{\rightleftarrows}} \boxed{i} \ \cdots$$

By equating the rate out of the set $\{i, i+1, \ldots\}$ to the rate into this set, we find the recurrence relations

$$\mu p_i = \frac{\lambda}{i} p_{i-1}, \ i = 1, 2, \ldots$$

These equations lead to

$$p_i = \frac{(\lambda/\mu)^i}{i!} p_0, \quad i \geq 1.$$

Using the normalizing equation $\sum p_i = 1$ we obtain

$$p_i = e^{-\lambda/\mu} \frac{(\lambda/\mu)^i}{i!}, \quad i \geq 0.$$

(b) By the PASTA property, the long-run fraction of arrivals that actually join the queue is

$$\sum_{i=0}^{\infty} p_i \frac{1}{i+1} = \frac{\mu}{\lambda}\left(1 - e^{-\lambda/\mu}\right).$$

The long-run average number of persons served per time unit is

$$\lambda \left[\frac{\mu}{\lambda}\left(1 - e^{-\lambda/\mu}\right)\right] = \mu(1 - p_0),$$

in agreement with the Little's formula.