MSG800/MVE170 Basic Stochastic Processes Fall 2010

Exercise Session 7, Thursday 9 December

Archetypical type-problems of typical type-exam-type

Archetypical type-problem of typical type-exam-type 1. Consider a Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space E and transition matrix P given by

$$E = \{0,1,2\} \qquad \text{and} \qquad P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix},$$

respectively, and let $E_i = \mathbf{E}\big\{\min\{n \geq 1 : X_n = i\}\,\big|\, X_0 = i\big\}$ for $i = 0,1,2$. Show that the chain has stationary distribution $[1/E_0\ \ 1/E_1\ \ 1/E_2]$.

Solution. By symmetry, the chain has stationary distribution $[1/3\ \ 1/3\ \ 1/3]$ and $E_0 = E_1 = E_2$, so it is enough to show that $E_0 = 3$. However, from state 0 it takes the chain one time-unit to move to one of the states 1 or 2. After that it takes the chain a geomteric distribution with parameter $p = 1/2$ to come back to state 0, the expected value of which is 2.

Archetypical type-problem of typical type-exam-type 2. Let $\{W(t)\}_{t\geq 0}$ be a Wiener process and $\lambda > 0$ a constant. Show that $\{W(\lambda t)\}_{t\geq 0}$ is also a Wiener process.

Solution. As $\{W(\lambda t)\}_{t\geq 0}$ starts at the value zero at time $t = 0$ (as W does), is zero-mean (as W is), and has independent increments (as W does), it only remains to check that $\{W(\lambda t)\}_{t\geq 0}$ has stationary normally distributed increments. This in turn follows from the fact that $W(\lambda t) - W(\lambda s)$ has the same distribution as $W(\lambda(t-s))$, which is $\mathrm{N}(0, \lambda(t-s))$-distributed

Archetypical type-problem of typical type-exam-type 3. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. Show that $\{\mathrm{e}^{-\lambda t} 2^{N(t)}\}_{t\geq 0}$ is a martingale wrt. the filtration F_s, $s \geq 0$, containing all information about the process values $\{N(r)\}_{r\leq s}$.

Solution. $\mathbf{E}\{\mathrm{e}^{-\lambda t} 2^{N(t)} | F_s\} = \mathrm{e}^{-\lambda t} 2^{N(s)} \mathbf{E}\{2^{N(t)-N(s)} | F_s\} =$ [independent increments] $= \mathrm{e}^{-\lambda t} 2^{N(s)} \mathbf{E}\{2^{N(t)-N(s)}\} = \mathrm{e}^{-\lambda t} 2^{N(s)} \mathbf{E}\{2^{\mathrm{Po}(\lambda(t-s))}\} = \mathrm{e}^{-\lambda t} 2^{N(s)} \sum_{k=0}^{\infty} 2^k (\lambda(t-s))^k$ $\mathrm{e}^{-\lambda(t-s)}/(k!) = \mathrm{e}^{-\lambda t} 2^{N(s)} \mathrm{e}^{\lambda(t-s)} = \mathrm{e}^{-\lambda s} 2^{N(s)}$ for $0 \leq s \leq t$.

Archetypical type-problem of typical type-exam-type 4. Let $\{e_n\}_{n\in\mathbb{Z}}$ be uncorrelated zero-mean and unit variance random variables (i.e., discrete time white noise). Find the autocorrelation function of the process $\{X_n\}_{n\in\mathbb{Z}}$ given by $X_n = e_n + e_{n-1}/2$.

Solution. We have $\mathbf{E}\{X_n^2\} = 1 + (1/2)^2 = 5/4$ and $\mathbf{E}\{X_n X_{n\pm1}\} = 1 \cdot (1/2) = 1/2$, while $\mathbf{E}\{X_n X_{n\pm k}\} = 0$ for $k > 1$.

Archetypical type-problem of typical type-exam-type 5. Recall that differentiating a random process $\{X(t)\}_{t\in\mathbb{R}}$ corresponds to processing the process through a linear system with frequency responce $H(\omega) = j\omega$. Show that the derivative of a WSS process X with autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}$ has autocorrelation function $R_{X'X'}(\tau) = 2\delta(\tau) - \mathrm{e}^{-|\tau|}$.

Solution. As X has PSD $S_{XX}(\omega) = 2/(1+\omega^2)$ it follows that X' has PSD $|H(\omega)|^2 \times S_{XX}(\omega) = 2\omega^2/(1+\omega^2) = 2 - S_{XX}(\omega)$, which is the Fourier-transform of $2\delta(\tau) - \mathrm{e}^{-|\tau|}$.

Archetypical type-problem of typical type-exam-type 6. Let $N(t)$ for $t \geq 0$ denote the total number of customers in a M/M/2/4 queuing system with exp(1)-distributed times between arriving customers as well as exp(1)-distributed service times. Assume that $N(0) = 0$ and let $\{T_n\}_{n=1}^{\infty}$ be the strictly increasing sequence of random times at which $\{N(t)\}_{t\geq0}$ changes its values, that is, $T_{n+1} = \min\{t > T_n : N(t) \neq N(T_n)\}$ for $n \in \mathbb{N}$, with the convention $T_0 = 0$. Find the transition matrix P for the Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space $E = \{0, 1, 2, 3, 4\}$ given by $X_n = N(T_n)$.

Solution. We have

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}:$$

Here the first two rows of P as well as the last one are obvious, while the entries of the third and fourth row follows from noting that the probability that an exp(1)-distributed interarrival time is bigger than two independent exp(1)-distributed service times is 1/3.