MSG800/MVE170 Basic Stochastic Processes

Written exam Saturday 25 August 2012 8.30 am - 12.30 am

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

GOOD LUCK!

Task 1. Consider a Markov chain $\{X(n) : n \geq 0\}$ with state space E, initial state probabilities $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = \{0,1\}, \quad \mathbf{p}(0) = [1/2 \;\; 1/2] \quad \text{and} \quad P = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix},$$

respectively. Find the probability $\mathbf{P}\{X(5)=1 \,|\, X(2)=1\}$. **(5 points)**

Task 2. Let $\{X(t) : t \geq 0\}$ be a zero-mean continuous-time normal random process with autocovariance fuction $\mathbf{Cov}\{X(s), X(t)\} = \min\{s,t\}$. Show that $X(t)$ is a self-similar process, which is to say that there exists a so called Hurst parameter $H > 0$ such that the processes $\{X(\lambda t) : t \geq 0\}$ and $\{\lambda^H X(t) : t \geq 0\}$ have the same n-th order distributions $\mathbf{P}\{X(\lambda t_1) \leq x_1, \ldots, X(\lambda t_n) \leq x_n\} = \mathbf{P}\{\lambda^H X(t_1) \leq x_1, \ldots, \lambda^H X(t_n) \leq x_n\}$ for any given constant $\lambda > 0$ and any $n \in \mathbb{N}$. **(5 points)**

Task 3. Let $\{Y_n : n \geq 0\}$ be a simple random walk given by $Y_n = \sum_{k=1}^{n} X_k$ where $X_1, X_2, \ldots$ are independent identically distributed $\{-1,1\}$-valued random variables with $\mathbf{P}\{X_k = -1\} = \mathbf{P}\{X_k = 1\} = \frac{1}{2}$. Consider the first exit time $T = \min\{n \geq 0 : Y_n \leq -a \text{ or } Y_n \geq b\}$ of $\{Y_n\}_{n=0}^{\infty}$ from the interval $(-a,b)$, where $a,b > 0$ are real constants. Find the probabilities $\mathbf{P}\{Y_T \leq -a\}$ and $\mathbf{P}\{Y_T \geq b\}$ that this exit is downwards and upwards, respectively. **(5 points)**

Task 4. Let $\{X(t) : t \in \mathbb{R}\}$ be a continuous-time wide-sense stationary random process with power spectral density $S_X(\omega)$. The derivative process of $X(t)$ is defined as $X'(t) = \lim_{h \to 0}(X(t+h) - X(t))/h$ whenever this limit is well-defined in a suitable sense. Show that the cross power spectral density between $X(t)$ and $X'(t)$ is given by $S_{XX'}(\omega) = j\omega\, S_X(\omega)$. **(5 points)**

Task 5. Let $\{X(t) : t \in \mathbb{R}\}$ be a continuous-time zero-mean wide-sense stationary random process with autocorrelation function $R_X(\tau) = \mathrm{e}^{-|\tau|}$ for $\tau \in \mathbb{R}$. Find the impulse response function $h(t)$ of the linear time invariant system with input $X(t)$ that has a zero-mean wide-sense stationary output process $\{Y(t):t\in\mathbb{R}\}$ such that $X(t)$ and $Y(t)$ has crosscorrelation function $R_{XY}(\tau)=\mathrm{e}^{-2|\tau|}$ for $\tau\in\mathbb{R}$. **(5 points)**

Task 6. Write a computer programme that by means of stochastic simulation finds an approximation of the variance of a typical waiting time $W(q)$ (in the queue) before service for a typical customer arriving to a steady-state M(1)/M(2)/1/2 queuing system. (In other words, the queuing system has exp(1)-distributed times between arrivals of new customers and exp(2)-distributed service times. Further, the system has one server and one queuing place.) **(5 points)**

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Solutions to written exam Saturday 25 August

Task 1. We have $\mathbf{P}\{X(5)=1\,|\,X(2)=1\} = (P^3)_{1,1}$ (i.e., the lower diagonal element of the third power of P), which by elementary matrix calculations equals 14/27.

Task 2. As $\{X(\lambda t):t\geq 0\}$ and $\{\lambda^H X(t):t\geq 0\}$ are both zero-mean normal processes their n-th order distributions agree if their auto-covariance functions do. These in turn are given by $\mathbf{Cov}\{X(\lambda s),X(\lambda t)\} = \min\{\lambda s,\lambda t\} = \lambda\min\{s,t\}$ and $\mathbf{Cov}\{\lambda^H X(s),\lambda^H X(t)\} = \lambda^{2H}\mathbf{Cov}\{X(s),X(t)\} = \lambda^{2H}\min\{s,t\}$, respectively, so that they agree for $H=\frac{1}{2}$.

Task 3. As $\{Y_n:n\geq 0\}$ is a martingale and T a stopping time that satisfy the conditions of the optional stopping theorem, writing $\lceil x\rceil$ for the smallest integer that is greater or equal to $x\in\mathbb{R}$, we have $0 = \mathbf{E}\{Y_0\} = \mathbf{E}\{Y_T\} = \mathbf{P}\{Y_T\leq -a\}\times(-\lceil a\rceil)+\mathbf{P}\{Y_T\geq b\}\times\lceil b\rceil = \mathbf{P}\{Y_T\leq -a\}\times(-\lceil a\rceil) + \big(1-\mathbf{P}\{Y_T\leq -a\}\big)\times\lceil b\rceil$, so that $\mathbf{P}\{Y_T\leq -a\} = \lceil b\rceil/(\lceil a\rceil+\lceil b\rceil)$ and $\mathbf{P}\{Y_T\geq b\} = 1-\mathbf{P}\{Y_T\leq -a\} = \lceil a\rceil/(\lceil a\rceil+\lceil b\rceil)$.

Task 4. As $R_{XX'}(\tau) = \mathbf{E}\big\{X(t)\lim_{h\to 0}(X(t+\tau+h)-X(t+\tau))/h\big\} = \lim_{h\to 0}\mathbf{E}\big\{X(t)(X(t+\tau+h)-X(t+\tau))/h\big\} = \lim_{h\to 0}(R_X(\tau+h)-R_X(\tau))/h = R_X'(\tau)$, we have $S_{XX'}(\omega) = \int_{-\infty}^{\infty}R_{XX'}(\tau)\,\mathrm{e}^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}R_X'(\tau)\,\mathrm{e}^{-j\omega\tau}\,d\tau = j\omega\int_{-\infty}^{\infty}R_X(\tau)\,\mathrm{e}^{-j\omega\tau}\,d\tau = j\omega\,S_X(\omega)$.

Task 5. As $S_X(\omega) = 1/(1+\omega^2)$ and $S_{XY}(\omega) = 4/(4+\omega^2)$ are related by $S_{XY}(\omega) = H(\omega)\,S_X(\omega)$, we have $H(\omega) = S_{XY}(\omega)/S_X(\omega) = 4\,(1+\omega^2)/(4+\omega^2) = 4-3\,S_{XY}(\omega)$ so that $h(t) = 4\,\delta(t)-3\,\mathrm{e}^{-2|t|}$.

Task 6.

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In[1]:= Clear[Reps,lambda,mu,i,Arr,Serv,W];
          {Reps,lambda,mu,W}={1000000,1,2,{}};
In[2]:= For[i=1,i<=Reps,i++,
          Arr=Random[ExponentialDistribution[lambda]];
          Serv=Random[ExponentialDistribution[mu]];
          If[Serv<Arr, AppendTo[W,0],
            AppendTo[W,Random[ExponentialDistribution[mu]]]]];
In[3]:= Variance[W]
Out[3]:= 0.141395
```