Alternative Solution to Problem 6.26

In this alternative Solution to Problem 6.26 we do not use the frequency alanlysis approach that is employed in the solution in Hsu's book, but instead use Equation 6.62 in the book together with direct calculation.

Accorsing to Equation 6.62 we have

$$R_Y(\tau) = h \star h(-\cdot) \star R_X(\tau).$$

Here we have

$$h(-\cdot) \star R_X(t) = \int_{-\infty}^{0} \mathrm{e}^{bu}\mathrm{e}^{-a|t-u|}\,du = \int_{0}^{\infty} \mathrm{e}^{-bu}\mathrm{e}^{-a|t+u|}\,du = \mathrm{e}^{-at}\int_{0}^{\infty} \mathrm{e}^{-(a+b)u}\,du = \frac{\mathrm{e}^{-at}}{a+b}$$

for $t \geq 0$, while $h(-\cdot) \star R_X(t)$ is equal to

$$\int_{0}^{\infty} \mathrm{e}^{-bu}\mathrm{e}^{-a|t+u|}\,du = \mathrm{e}^{at}\int_{0}^{-t} \mathrm{e}^{(a-b)u}\,du + \mathrm{e}^{-at}\int_{-t}^{\infty} \mathrm{e}^{-(a+b)u}\,du = \ldots = \frac{2a\,\mathrm{e}^{bt}}{a^2-b^2} - \frac{\mathrm{e}^{at}}{a-b}$$

for $t < 0$. For $\tau \geq 0$ this gives that $h \star h(-\cdot) \star R_X(\tau)$ is equal to

$$\int_{-\infty}^{\infty} h(\tau-t)\,(h(-\cdot)\star R_X)(t)\,dt = \int_{0}^{\tau} \mathrm{e}^{-b(\tau-t)}\,\frac{\mathrm{e}^{-at}}{a+b}\,dt + \int_{-\infty}^{0} \mathrm{e}^{-b(\tau-t)}\left(\frac{2a\,\mathrm{e}^{bt}}{a^2-b^2} - \frac{\mathrm{e}^{at}}{a-b}\right)dt,$$

where

$$\int_{0}^{\tau} \mathrm{e}^{-b(\tau-t)}\,\frac{\mathrm{e}^{-at}}{a+b}\,dt = \ldots = \frac{\mathrm{e}^{-b\tau} - \mathrm{e}^{-a\tau}}{a^2-b^2}$$

and

$$\int_{-\infty}^{0} \mathrm{e}^{-b(\tau-t)}\left(\frac{2a\,\mathrm{e}^{bt}}{a^2-b^2} - \frac{\mathrm{e}^{at}}{a-b}\right)dt = \ldots = \mathrm{e}^{-b\tau}\left(\frac{a/b}{a^2-b^2} - \frac{1}{a^2-b^2}\right).$$

Consequently

$$R_Y(\tau) = \frac{a/b}{a^2-b^2}\,\mathrm{e}^{-b\tau} - \frac{1}{a^2-b^2}\,\mathrm{e}^{-a\tau}$$

for $\tau \geq 0$, so that by symmetry

$$R_Y(\tau) = \frac{a/b}{a^2-b^2}\,\mathrm{e}^{-b|\tau|} - \frac{1}{a^2-b^2}\,\mathrm{e}^{-a|\tau|}$$

for $\tau \in \mathbb{R}$.