Addendum on some Basic Results in Gambling Theory

Consider the martingale as well as Markov chain $\{M_n\}_{n=0}^{\infty}$ given by $M_n = m_0+\sum_{i=1}^{n} X_i$ where $\{X_i\}_{i=1}^{\infty}$ are iid. (independent identically distributed) random variables with $\mathbf{P}\{X_i = -1\} = \mathbf{P}\{X_i = 1\} = 1/2$ (a so called Rademacher distribution): This is the simplest possible (non-trivial) example of a fair gambling process with initial fortune m_0.

For the so called n-step transition probability $p_{ij}^{(n)} = \mathbf{P}\{M_n = j\,|\,M_0 = i\}$ we then have $p_{jj}^{(n)} = 0$ for $n = 2k+1$ odd and

$$p_{jj}^{(n)} = \binom{2k}{k}(1/2)^k(1/2)^k \quad \text{for } \ n = 2k \ \text{ even}$$

[=the probability that a $\mathrm{Bin}(2k, 1/2)$-distributed random variable is equal to k (=the probability that half of the n steps go upwards and the other half of them downwards)].

It is a theorem in Markov theory that a state j is reccurent (that is, it is certain to get back to the state eventually if starting in it) if $\sum_{n=1}^{\infty} p_{jj}^{(n)} = \infty$, while j is transient (that is, it is not certain to get back to the state if starting in it) if $\sum_{n=1}^{\infty} p_{jj}^{(n)} < \infty$[1].

Now, according to Stirling's formula we have $k! \sim \sqrt{2\pi k}\, k^k\, \mathrm{e}^{-k}$ as $k \to \infty$ (where $\sim$ means "behaves asymptotically like"). Hence

$$\binom{2k}{k}(1/2)^k(1/2)^k = \frac{(2k)!\,(1/2)^{2k}}{(k!)(k!)} \sim \frac{\sqrt{4\pi k}\,(2k)^{2k}\,\mathrm{e}^{-2k}\,(1/2)^{2k}}{(\sqrt{2\pi k}\,k^k\,\mathrm{e}^{-k})^2} = \frac{1}{\sqrt{\pi k}} \quad \text{as } \ k \to \infty,$$

so that $\sum_{n=1}^{\infty} p_{jj}^{(n)} = \sum_{k=1}^{\infty} p_{jj}^{(2k)} = \infty$: Consequently, $\{M_n\}_{n=0}^{\infty}$ is reccurent!

Now consider the probability $\mathbf{P}\{M_n = j \ \text{ for some } \ n \geq 1\} = \mathbf{P}\{M_n = j \ \text{ for some } \ n \geq 1\,|\,M_0 = m_0\}$ to reach a state j eventually with initial fortune m_0. If we write $j = m_0 + N$ for a suitable integer N we see that $\mathbf{P}\{M_N = j\} = (1/2)^{|N|}$. As the chain is reccurent we have after each (certain!) return of the chain to the initial state the probability $(1/2)^{|N|}$ to reach j in N steps. And by repeating this experiment add infinum (at each return to m_0) we see that we must have $\mathbf{P}\{M_n = j \ \text{ for some } \ n \geq 1\} = 1$.

To prove that the expected value of the time considered in the previous paragraph (found to be finite with probability 1) it takes the chain to reach the state j when

[1]Proof: Let $P_j(z) = \sum_{n=0}^{\infty} p_{jj}^{(n)} z^n$ and $F_j(z) = \sum_{n=0}^{\infty} f_{jj}^{(n)} z^n$ for $|z| < 1$ with $f_{jj}^{(n)} = \mathbf{P}\{M_n = j,\, M_{n-1} \neq j, \ldots, M_1 \neq j\,|\,M_0 = j\}$ and $f_{jj}^{(0)} = 0$. Then $p_{jj}^{(n)} = \sum_{k=0}^{n} f_{jj}^{(k)} p_{jj}^{(n-k)}$ (a discrete convolution) for $n \geq 1$, so that $P_j(z) = 1+\sum_{n=1}^{\infty} p_{jj}^{(n)} z^n = 1+\sum_{n=1}^{\infty}\sum_{k=0}^{n} f_{jj}^{(k)} p_{jj}^{(n-k)} z^n = 1+\sum_{n=0}^{\infty}\sum_{k=0}^{n} f_{jj}^{(k)} p_{jj}^{(n-k)} z^n = 1 + \sum_{k=0}^{\infty}(\sum_{n=k}^{\infty} p_{jj}^{(n-k)} z^{n-k})\, f_{jj}^{(k)} z^k = 1 + P_j(z)F_j(z)$, so that $\mathbf{P}\{M_n = j \ \text{ for some } \ n \geq 1\,|\,M_0 = j\} = \sum_{n=1}^{\infty} f_{jj}^{(n)} = \lim_{z\uparrow 1} F_j(z) = \lim_{z\uparrow 1}(P_j(z)-1)/P_j(z)) = 1$ (=reccurence) if $\sum_{n=1}^{\infty} p_{jj}^{(n)} = \lim_{z\uparrow 1} P_j(z) = \infty$ while the same probability is strictly less than 1 (=transience) if $\sum_{n=1}^{\infty} p_{jj}^{(n)} = \lim_{z\uparrow 1} P_j(z) < \infty$.

starting at the state m_0 is always infinite we may without loss of generality consider the special case when $m_0 = 0$ and $j = 1$, as it is clear that if the expectation is infinite in that special case, then it will be infinite for all other cases as well (as the chain then has a longer way to travel to its goal). Now consider the modified chain $\{\bar{M}_n\}_{n=0}^{\infty}$ with state space E, initial probability $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = [\dots\ -2\ -1\ 0\ 1], \quad \mathbf{p}(0) = [\dots\ 0\ 0\ 0\ 1] \quad \text{and} \quad P = \begin{bmatrix} \dots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 1/2 & 0 & 0 \\ \dots & 1/2 & 0 & 1/2 & 0 \\ \dots & 0 & 1/2 & 0 & 1/2 \\ \dots & 0 & 0 & 1 & 0 \end{bmatrix},$$

respectively. Clearly, it is enough to show that we have $\mathbf{E}\{T_1\} = \infty$ for the reccurence time $T_1 = \min\{n \geq 1 : \bar{M}_n = 1\}$ as the expectation we are interested in for the original chain must be $\mathbf{E}\{T_1\} - 1$. This in turn we can establish by means of proving that the chain doesn't have a stationary distribution, as in the presence of a stationary distribution $\pi = [\dots\ -\pi_{-2}\ \pi_{-1}\ \pi_0\ \pi_1]$ we must have $\mathbf{E}\{T_1\} = 1/\pi_1$ where all elements of π (and in particular π_1) must be strictly positive as a consequence of the fact that all states communicate with each other. However, the equation system $\pi = \pi P$ to find the stationary distribution (if it exists) spells out as

$$(1/2)\,\pi_{n-1} + (1/2)\,\pi_{n+1} = \pi_n \quad \text{for } n < 0, \quad (1/2)\,\pi_{-1} + \pi_1 = \pi_0 \quad \text{and} \quad (1/2)\,\pi_0 = \pi_1,$$

which in turn has unique solution $\pi = [\dots\ \pi_0\ \pi_0\ \pi_0\ (1/2)\,\pi_0]$. This however can never be a probability distribution (as it cannot sum up to 1), and we are done!

Så de så![2]

[2] Remark of the type setter.