MSG800/MVE170 Basic Stochastic Processes Fall 2010

Written exam Monday 13 December 8.30 am - 1.30 pm

Teacher and jour: Patrik Albin.

Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Task 1. Recall the computer problem from Exercise Session 2 to find the expected value $\mathbf{E}\{T\}$ of the time $T = \min\{n \in \mathbb{N} : X_n = 2\}$ it takes the Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space E, initial distribution $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = \{0,1,2\}, \quad \mathbf{p}(0) = [1\ \ 0\ \ 0] \quad \text{and} \quad P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix},$$

respectively, to reach the state 2. In order to treat that problem analytically we notice that $T+1$ has the same distribution as the reccurence time $T_2 = \min\{n \geq 1 : \hat{X}_n = 2\}$ for the Markov chain $\{\hat{X}_n\}_{n=0}^{\infty}$ with

$$\hat{E} = \{0,1,2\}, \quad \hat{\mathbf{p}}(0) = [0\ \ 0\ \ 1] \quad \text{and} \quad \hat{P} = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}.$$

Writing $\pi = [\pi_0\ \ \pi_1\ \ \pi_2]$ for the stationary distribution of $\{\hat{X}_n\}_{n=0}^{\infty}$ theory says (as well as does heuristics) that $\pi_2 = 1/\mathbf{E}\{T_2\}$. Use this to calculate $\mathbf{E}\{T\}$. **(5 points)**

Task 2. Let $\{W(t)\}_{t\geq 0}$ be a Wiener process with $\mathbf{E}\{W(1)^2\} = 2$. Express the probability $\mathbf{P}\{W(1)+W(2) > 3\}$ in terms of the standard normal cummulative probability distribution function $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2}\, dy$. **(5 points)**

Task 3. Write a computer programme in some easy to understand pseudo-code that determines by means of simulations an approximative value for the probability $\mathbf{P}\{X(t) > t \text{ for some } t \in [0,10]\}$ for a unit rate Poisson process $\{X(t)\}_{t\geq 0}$. **(5 points)**

Task 4. Can $R_{XX}(\tau) = \cos(\tau)$ be the autocorrelation function of a WSS process $\{X(t)\}_{t\in\mathbb{R}}$? Can $R_{XX}(\tau) = \sin(\tau)$ be the autocorrelation function of a WSS process $\{X(t)\}_{t\in\mathbb{R}}$? Can $\mu_X(t) = \cos(t)$ be the mean function of a WSS process $\{X(t)\}_{t\in\mathbb{R}}$? Can

$S_{XX}(\omega) = \cos(\omega)$ be the power spectral density (PSD) of a WSS process $\{X(t)\}_{t\in\mathbb{R}}$? Can $S_{XX}(\omega) = 1$ be the PSD of a WSS process $\{X(t)\}_{t\in\mathbb{R}}$? **(5 points)**

Task 5. A WSS random signal $\{X(t)\}_{t\in\mathbb{R}}$ with PSD $S_{XX}(\omega)$ is transmitted on a noisy channel where it is disturbed by an additive zero-mean WSS random noise $\{N(t)\}_{t\in\mathbb{R}}$ that is independent of the signal X and has PSD $S_{NN}(\omega)$. The recived signal $Y(t) = X(t)+N(t)$ is input to a linear system (/filter) with output signal $Z(t)$ that has frequency response $H(\omega) = S_{XX}(\omega)/(S_{XX}(\omega) + S_{NN}(\omega))$. Express the mean-square deviation $\mathbf{E}\{(Z(t) - X(t))^2\}$ in terms of S_{XX} and S_{NN}. **(5 points)**

Task 6. Customers arrive at the express checkout lane in a supermarket according to a Poisson process with rate 15 per hour. The time to check out a customer is an exponential random variable with mean 2 minutes. Under natural independence assumptions (between arrival times and check out times), find the average number of customers present in the supermarket at steady-state. **(5 points)**

$\boxed{\textbf{Good Luck!}}$

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Solutions to written exam Monday 13 December

Task 1. We find π as the PMF on E that solves the equation $\pi\hat{P} = \pi$. As this gives $\pi = [\frac{2}{5}\ \ \frac{2}{5}\ \ \frac{1}{5}]$ it follows that $\mathbf{E}\{T\} = \mathbf{E}\{T_2\} - 1 = 1/\pi_2 - 1 = 5 - 1 = 4$.

Task 2. $\mathbf{P}\{W(1)+W(2) > 3\} = 1 - \mathbf{P}\{\mathrm{N}(0,\sigma^2) \le 3\} = 1 - \Phi(3/\sigma)$, where $\sigma^2 = \mathbf{E}\{(W(1)+W(2))^2\} = R_{WW}(1,1) + R_{WW}(2,2) + 2\,R_{WW}(1,2) = 2+4+4 = 10$.

Task 3.
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rep=100000;For[i=1;count=0,i<=rep,i++,t=0;X=0;succ=0;
    While[(t<=10)&&(succ=0),t=t+Random[ExponentialDistribution[1]];
    X=X+1;If[X>t,succ=1]];count=count+succ];N[count/rep]
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Task 4. Yes, see Problem 5.20. No, as it does not hold that $|R_{XX}(\tau)| \le R_{XX}(0)$. No, as WSS processes have constant mean. No, as $S_{XX}(\omega)$ is not non-negative. Yes or no depending on what you mean, as this is the PSD of continuous time white noise (which does not really exist), see Section 6.4.

Task 5. Writing h for the impulse response of the filter and $\star$ for convolution the fact that $h\star N$ is zero-mean and independent of X (as N is) readily gives that $\mathbf{E}\{(Z(t)-X(t))^2\} = \mathbf{E}\{((h\star X)(t) + (h\star N)(t) - X(t))^2\} = \mathbf{E}\{(((h-\delta)\star X)(t) + (h\star N)(t))^2\} = \mathbf{E}\{((h-\delta)\star X)(t)^2 + (h\star N)(t)^2\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[|H(\omega)-1|^2 S_{XX}(\omega) + |H(\omega)|^2 S_{NN}(\omega)]\,d\omega = \ldots = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)S_{NN}(\omega)/(S_{XX}(\omega)+S_{NN}(\omega))\,d\omega$.

Task 6. This is Problem 9.21 a, see also Problem 9.4.