MVE170/MSG800 Basic Stochastic Processes

Written exam Thursday 28 April 2011 8.30 am–1.30 pm

Teacher and jour: Patrik Albin, telephone 070 6945709.

Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG, and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

GOOD LUCK!

Task 1. State a problem that argubly is very hard to solve analytically, but that can be solved in a rather straightforward manner by means of computer simulations. Supply a computer program (e.g., in terms of a flow chart or so called "pesudo-code") that solves the problem. **(5 points)**

Task 2. Give an example of a discrete-parameter Markov chain $\{X(n),\, n \geq 0\}$ that has both periodic and aperiodic states. **(5 points)**

Task 3. Let $\{X(t),\, t \geq 0\}$ be a Wiener process. Show that the process $\{Y(t),\, t \in \mathbb{R}\}$ given by $Y(t) = \mathrm{e}^{-t} X(\mathrm{e}^{2t})$ is a stationary process. **(5 points)**

Task 4. Consider a discrete time stochastic process $\{X(n),\, n \geq 0\}$ given by $X(0) = 1$ and $X(n) = Y_1 \cdot \ldots \cdot Y_n$ for $n \geq 1$, where $Y_1, Y_2, \ldots$ are independent random variables. Under which additional conditions on the random variables $Y_1, Y_2, \ldots$ is $\{X(n),\, n \geq 0\}$ a Markov process? Under which additional conditions on the random variables $Y_1, Y_2, \ldots$ is $\{X(n),\, n \geq 0\}$ a martingale? **(5 points)**

Task 5. Find the impulse response $h(t)$ of a continuous time LTI system the output $Y(t)$ of which is WSS with autocorrelation function $R_Y(\tau) = 1/(1+\tau^2)$ when the input is continuous time white noise $W(t)$. **(5 points)**

Task 6. What proportions of the total time does a M/M/1/2 queueing system with traffic intensity $\rho = 1$ spend in its three different states 0, 1 and 2 (empty system, server occupied but queue empty, and both server and queue occupied, respectively)? Is there any difference between the times that are spent in each state before switcing to the next state? **(5 points)**

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Solutions to written exam Thursday 28 April 2011

Task 1. See, e.g., the computational tasks of the course.

Task 2. For example the chain with state space E and transition probability matrix P given by

$$E = \{0,1,2,3\} \qquad \text{and} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix},$$

respectively, where the states $\{0,1\}$ have period 2 while the states $\{2,3\}$ are aperiodic.

Task 3. As X is a normal process so is Y, and so Y is stationary if it is WSS. As X is zero-mean so is Y. As X has autocorrelation function $R_X(s,t) = \sigma^2 \min(s,t)$ we have $R_Y(s,t) = E[Y(s)Y(t)] = E[\mathrm{e}^{-s}X(\mathrm{e}^{2s})\mathrm{e}^{-t}X(\mathrm{e}^{2t})] = \mathrm{e}^{-(s+t)}R_X(\mathrm{e}^{2s},\mathrm{e}^{2t}) = \mathrm{e}^{-(s+t)}\sigma^2 \min(\mathrm{e}^{2s},\mathrm{e}^{2t}) = \sigma^2\mathrm{e}^{-|t-s|}$. Hence Y is WSS.

Task 4. It is easy to see that $X(n)$ is always a Markov process, but is a martingale if and only if $E[Y_n] = 1$ for $n \geq 1$.

Task 5. As $S_W(\omega) = \sigma^2$ and $\pi\,\mathrm{e}^{-|\omega|} = S_Y(\omega) = |H(\omega)|^2 S_W(\omega) = \sigma^2|H(\omega)|^2$, it will work to take $H(\omega) = \sqrt{\pi}\,\mathrm{e}^{-|\omega|/2}/\sigma$ which means that $h(t) = (\sigma\sqrt{\pi}\,((1/2)^2+t^2))^{-1}$.

Task 6. Sending $\rho \to 1$ in the formula $p_n = (1-\rho)\,\rho^n/(1-\rho^{K+1}) = (1-\rho)\,\rho^n/(1-\rho^2)$ for $n = 0,1,2$ we get $p_0 = p_1 = p_2 = 1/3$. Hence an equal proportion of time 1/3 is spent in each state. From this it follows readily that the time spent in state 1 before switching to the next state must be half as long as that spent in states 0 and 2, as state 1 will be visited twice as many times as the latter two states.