MVE170/MSG800 Basic Stochastic Processes Fall 2010

Written exam Friday 19 August 2011 8.30 am–1.30 pm

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12, 18, 21 and 24 points for grades 3/G, 4, VG and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

GOOD LUCK!

Task 1. Consider a discrete-parameter Markov chain $\{X_n,\, n \geq 0\}$ with state space E, transition probability matrix P and initial-state probabilities $\mathbf{p}(0)$ given by

$$E = \{0,1,2,3\}, \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{p}(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

How can one simulate an observation X_{10} of the chain at time $n = 10$? **(5 points)**

Task 2. Let $\{X(t),\, t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Is it possible to find a deterministic (=non-random) function $f : [0,\infty) \to \mathbb{R}$ such that the process $Y(t) = f(t)\, \mathrm{e}^{X(t)}$, $t \geq 0$, is a continuous time martingale with respect to the filtration $F_t = \sigma(X(s),\, s \leq t)$, $t \geq 0$, generated by the history of the Poisson process? **(5 points)**

Task 3. A continuous time random process $\{Y(t),\, t \in \mathbb{R}\}$ is defined by $Y(t) = A\, X(t) \times \cos(\omega_c t + \Theta)$ for $t \in \mathbb{R}$, where $A, \omega_c \in \mathbb{R}$ are constants, Θ is a random variable that is uniformly distributed over the interval $(-\pi, \pi)$ and $\{X(t),\, t \in \mathbb{R}\}$ is a zero-mean WSS random process with power spectral density $S_X(\omega)$ that is independent of Θ. Show that $Y(t)$ is a WSS process and find its power spectral density $S_Y(\omega)$. **(5 points)**

Task 4. A post office has three clerks serving at the counter. Customers arrive according to a Poisson process at the rate of 30 per hour, and arriving customers are asked to form a single queue. The service times for customers are exponetial distributed random variables with mean 3 minutes. Find (a) the probability that all the clerks will be busy (at any given specific time), (b) the average number of customers in the queue, and (c) the average length of time customers have to spend in the post office. **(5 points)**

Task 5. Let $\{X(t),\, t \geq 0\}$ be a WSS continuous time martingale. Show that $E\big((X(t)-X(s))^2\big) = E(X(t)^2) - E(X(s)^2)$ for $0 \leq s \leq t$. Use this identity to show that $X(t) = X(0)$ (with probability 1) for $t \geq 0$. **(5 points)**

Task 6. Let $\{X(n),\, n \in \{1,2\}\}$ be a stationary zero-mean Gaussian process with autocorrelation function $R_X(0) = 1$ and $R_X(\pm 1) = \rho$ for a constant $\rho \in [-1,1]$. Show that for each $x \in \mathbb{R}$ it holds that $\max_{n\in\{1,2\}} P(X(n) > x) \leq P\big(\max_{n\in\{1,2\}} X(n) > x\big)$. Are there any values of ρ for which this inequality becomes an equality? **(5 points)**

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Solutions to written exam Friday 19 August 2011

Task 1. See, e.g., the computational tasks of the course.

Task 2. By independence and stationarity of increments together with basic rules for conditional expectations we have $E\big(\mathrm{e}^{X(t)}\big|F_s\big) = E\big(\mathrm{e}^{X(t)-X(s)}\mathrm{e}^{X(s)}\big|F_s\big) = E\big(\mathrm{e}^{X(t)-X(s)}\big| F_s\big)\,\mathrm{e}^{X(s)} = E\big(\mathrm{e}^{X(t)-X(s)}\big)\,\mathrm{e}^{X(s)} = E(\mathrm{e}^{X(t-s)})\,\mathrm{e}^{X(s)} = \sum_{k=0}^{\infty}\mathrm{e}^k\,((\lambda(t-s))^k/(k!))\,\mathrm{e}^{-\lambda(t-s)} \times \mathrm{e}^{X(s)} = \mathrm{e}^{(\mathrm{e}-1)\lambda(t-s)}\,\mathrm{e}^{X(s)}$. Hence it is easy to see that $\{Y(t),\, t\geq 0\}$ is a martingale when $f(t) = \mathrm{e}^{-(\mathrm{e}-1)\lambda t}$.

Task 3. This is supplemantary problem 6.53 in Hsu's book listed as a problem for own work during the course.

Task 4. This is supplementary problem 9.26 in Hsu's book, which in turn is solved by insertion in known results from M/M/s queueing theory with $s=3$, $\lambda=30$/hour and $\mu=20$/hour.

Task 5. We have $E\big((X(t)-X(s))^2\big) = E(X(t)^2)+E(X(s)^2)-2\,E(X(t)X(s)) = E(X(t)^2)+E(X(s)^2)-2\,E\big(E(X(t)|F_s)\,X(s)\big) = E(X(t)^2)+E(X(s)^2)-2\,E(X(s)^2) = E(X(t)^2)-E(X(s)^2)$ for $0\leq s\leq t$. Further, we have $E(X(t)^2) = E(X(0)^2)$ as $X(t)$ is WSS, giving $E\big((X(t)-X(0))^2\big) = E(X(t)^2)-E(X(0)^2) = 0$ so that $X(t) = X(0)$.

Task 6. We have $\max_{n\in\{1,2\}} P(X(n)>x) = P(X(1)>x) = P(X(2)>x) = 1-\Phi(x)$ and $P\big(\max_{n\in\{1,2\}} X(n)>x\big) = P\big(\{X(1)>x\}\cup\{X(2)>x\}\big) = P(X(1)>x)+P(X(2) > x) - P\big(\{X(1)>x\}\cap\{X(2)>x\}\big) = 2\,(1-\Phi(x)) - P\big(\{X(1)>x\}\cap\{X(2)>x\}\big)$. Here $P\big(\{X(1)>x\}\cap\{X(2)>x\}\big) = P(X(1)>x) = P(X(2)>x) = 1-\Phi(x)$ when $\rho=1$ (so that $X(1)$ and $X(2)$ are perfectly positively correlated), giving equality in the desired inequality, while by conditioning $P\big(\{X(1)>x\}\cap\{X(2)>x\}\big) < P(X(1) > x) = P(X(2)>x) = 1-\Phi(x)$ for $\rho<1$, giving strict inequality in the desired inequality.