MSG800/MVE170 Basic Stochastic Processes Fall 2011

Written exam Monday 12 December 2011 2 pm – 6 pm

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

GOOD LUCK!

Task 1. Consider a homogeneous Markov chain $\{X_n,\, n\geq 0\}$ with state space (/possible values) E, initial state probability vector $\mathbf{p}(0)$ and transition matrix P given by

$$E=\{0,1\}, \qquad \mathbf{p}(0)=\left[\tfrac{\beta}{\alpha+\beta}\ \ \tfrac{\alpha}{\alpha+\beta}\right] \qquad \text{and} \qquad P=\begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix},$$

respectively for some constants $\alpha,\beta\in(0,1]$. Show that $\hat{\mathbf{p}}=\mathbf{p}(0)$ is a stationary distribution for the Markov chain. Find $\mathbf{E}\{X_n\}$ and $\mathbf{Var}\{X_n\}$ for $n\geq 0$. **(5 points)**

Task 2. Let $\{X(t),\, t\in\mathbb{R}\}$ be a WSS process and $\alpha>0$ a constant. Show that $\{X(\alpha t),\, t\in\mathbb{R}\}$, $\{X(t-\alpha),\, t\in\mathbb{R}\}$ and $\{X(-t),\, t\in\mathbb{R}\}$ are also WSS processes. **(5 points)**

Task 3. Consider the Markov chain $\{X_n,\, n\geq 0\}$ in Task 1 with $\alpha=\beta=\frac{1}{3}$ and with $\mathbf{p}(0)$ changed to $[1\ \ 0]$. Write a computer programme that by means of stochastic simulation finds an approximative value of the expected value $\mathbf{E}\{T\}$ of the random time $T=\min\{n\geq 10 : X_n=0 \text{ and } X_{n+1}=1\}$. **(5 points)**

Task 4. Let $\{X(t),\, t\geq 0\}$ be a Wiener process with $\mathbf{E}\{X(1)^2\}=1$. Show that $\{X(t)^2-t,\, t\geq 0\}$ is a martingale with respect to the information $F_t=\{X(s),\, s\in[0,t]\}$ obtained by observing the Wiener process up to time t. **(5 points)**

Task 5. Find the variance $\mathbf{Var}\{Y(t)\}$ of the output $Y(t)$ from a continuous-time LTI system with impulse response $h(t)=\mathrm{e}^{-t}$ for $t\geq 0$ and $h(t)=0$ for $t<0$ and with a zero-mean WSS input process $\{X(t),\, t\in\mathbb{R}\}$ with auto-correlation function $R_X(\tau)=\mathrm{e}^{-2|\tau|}$ for $\tau\in\mathbb{R}$. **(5 points)**

Task 6. Consider a M/M/1/2 queueing system with mean arrival rate $\lambda>0$ and mean service rate $\mu>0$. Find the average length of an idle period (that is, a period during which the server is idle) and the average lenght of a busy period (that is, a period during which the server is busy). **(5 points)**

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Solutions to written exam Monday 12 December

Task 1. As $\hat{\mathbf{p}} = \mathbf{p}(0)$ satisfies $\hat{\mathbf{p}}P = \hat{\mathbf{p}}$ it follows that $\hat{\mathbf{p}} = \mathbf{p}(0)$ is a stationary distribution. As $\mathbf{p}(n) = \mathbf{p}(0)P^n = (\mathbf{p}(0)P)\,P^{n-1} = \mathbf{p}(0)P^{n-1} = \ldots = \mathbf{p}(0)$ for $n \geq 0$ we have $\mathbf{E}\{X_n\} = \sum_{k=0}^{1} k\,\mathbf{P}\{X_n = k\} = \sum_{k=0}^{1} k\,\mathbf{p}(n)_k = \sum_{k=0}^{1} k\,\mathbf{p}(0)_k = \frac{\alpha}{\alpha+\beta}$ and similarly $\mathbf{E}\{X_n^2\} = \sum_{k=0}^{1} k^2\,\mathbf{p}(0)_k = \frac{\alpha}{\alpha+\beta}$ so that $\mathbf{Var}\{X_n\} = \mathbf{E}\{X_n^2\} - (\mathbf{E}\{X_n\})^2 = \ldots = \frac{\alpha\beta}{(\alpha+\beta)^2}$.

Task 2. Clearly, $X(\alpha t)$, $X(t-\alpha)$ and $X(-t)$ all have the same constant (time-invariant) mean μ_X as has $X(t)$ while $\mathbf{E}\{X(\alpha t)X(\alpha(t+\tau))\} = R_X(\alpha\tau)$, $\mathbf{E}\{X(t-\alpha)X(t+\tau-\alpha)\} = R_X(\tau)$ and $\mathbf{E}\{X(-t)X(-(t+\tau))\} = R_X(-\tau) = R_X(\tau)$ do not depend on $t \in \mathbb{R}$.

Task 3.

```
Clear[i, Time, Rep, X, Y];
For[i=1; Time=0; Rep=1000000000000000000000000000, i<=Rep, i++, X=0;
    For[j=1, j<=10, j++,
        If[Random[UniformDistribution[{0,1}]]<=1/3, X=1-X]];
        If[Random[UniformDistribution[{0,1}]]<=1/3, Y=1-X, Y=X];
    While[{X,Y}!={0,1}, Time=Time+1; X=Y;
        If[Random[UniformDistribution[{0,1}]]<=1/3, Y=1-X]]];
N[9+Time/Rep]
```

Task 4. $\mathbf{E}\{X(t)^2 - t|F_s\} = \mathbf{E}\{(X(t)-X(s))^2|F_s\} + 2\,\mathbf{E}\{(X(t)-X(s))\,X(s)|F_s\} + \mathbf{E}\{X(s)^2|F_s\} - t = \mathbf{E}\{(X(t)-X(s))^2\} + 2\,\mathbf{E}\{X(t)-X(s)|F_s\}\,X(s) + X(s)^2 - t = (t-s) + 2\,\mathbf{E}\{X(t)-X(s)\}\,X(s) + X(s)^2 - t = (t-s) + 0 + X(s)^2 - t = X(s)^2 - s$ for $s \leq t$ by independent increments of the Wiener process and (5.73)-(5.75) in Hsu's book.

Task 5. Since $S_X(\omega) = \frac{4}{4+\omega^2}$ and $H(\omega) = \frac{1}{1+j\omega}$ we have $S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \frac{4}{(1+\omega^2)\,(4+\omega^2)} = \frac{4/3}{1+\omega^2} - \frac{4/3}{4+\omega^2}$, so that $R_Y(\tau) = \frac{2}{3}\,\mathrm{e}^{-|\tau|} - \frac{1}{3}\,\mathrm{e}^{-2|\tau|}$. As $Y(t)$ is zero-mean this gives $\mathbf{Var}\{Y(t)\} = \mathbf{E}\{Y(t)^2\} = R_Y(0) = \frac{1}{3}$.

Task 6. The average idle time is the average of an exponential distribution with parameter λ which is $1/\lambda$. The average busy period B is the average time spent in state 1 plus the probability that the next state is two times the average time spent in two plus B, which in turn is $1/(\lambda+\mu)$ (i.e., the mean of the minimum of two independent exponentially distributed waiting time with parameters λ and μ, respectively, which equals the mean of an exponential distribution with parameter $\lambda+\mu$) plus $\lambda/(\lambda+\mu)$ times $1/\mu + B$. Hence we have $B = 1/(\lambda+\mu) + (\lambda/(\lambda+\mu)) \times (1/\mu + B)$, so that $B = (\lambda+\mu)/\mu^2$.