MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 26 August 2013 8.30–12.30 am

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Good Luck!

Task 1. Consider a homogeneous Markov chain $\{X_n,\, n\geq 0\}$ with state space (/possible values) E, initial state probability vector $\mathbf{p}(0)$ and transition matrix P given by

$$E=\{0,1\}, \qquad \mathbf{p}(0)=\left[\tfrac{\beta}{\alpha+\beta}\;\; \tfrac{\alpha}{\alpha+\beta}\right] \qquad \text{and} \qquad P=\begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix},$$

respectively for some constants $\alpha,\beta\in(0,1]$. Show that $\hat{\mathbf{p}}=\mathbf{p}(0)$ is a stationary distribution for the Markov chain. Find $\mathbf{E}\{X_n\}$ and $\mathbf{Var}\{X_n\}$ for $n\geq 0$. **(5 points)**

Task 2. Let $\{X(t):t\geq 0\}$ be a Poisson process with rate (/intensity) $\lambda>0$. Show that the process $\{M(t):t\geq 0\}$ given by $M(t)=(X(t)-\lambda t)^2-\lambda t$ for $t\geq 0$ is a martingale with respect to the knowledge (of the σ-field) F_t of all historic process values of the Poisson process. **(5 points)**

Task 3. Let $\{W(t)\}_{t\geq 0}$ be a Wiener process with $\mathbf{E}\{W(1)^2\}=2$. Express the probability $\mathbf{P}\{W(1)+W(2)>3\}$ in terms of the standard normal cummulative probability distribution function $\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-y^2/2}\,dy$. **(5 points)**

Task 4. Let $\{X(t):t\in\mathbb{R}\}$ be a continuous-time zero-mean wide-sense stationary random process with autocorrelation function $R_X(\tau)=\mathrm{e}^{-|\tau|}$ for $\tau\in\mathbb{R}$. Find the impulse response function $h(t)$ of the linear time invariant system with input $X(t)$ that has a zero-mean wide-sense stationary output process $\{Y(t):t\in\mathbb{R}\}$ such that $X(t)$ and $Y(t)$ has crosscorrelation function $R_{XY}(\tau)=\mathrm{e}^{-2|\tau|}$ for $\tau\in\mathbb{R}$. **(5 points)**

Task 5. A post office has three clerks serving at the counter. Customers arrive according to a Poisson process at the rate of 30 per hour, and arriving customers are asked to form a single queue. The service times for customers are exponetial distributed random variables with mean 3 minutes. Find (a) the probability that all the clerks will be busy (at any given specific time), (b) the average number of customers in the queue, and (c) the average length of time customers have to spend in the post office. **(5 points)**

Task 6. State a problem that argubly is very hard to solve analytically, but that can be solved in a rather straightforward manner by means of computer simulations. Supply a computer program (e.g., in terms of a flow chart or so called "pesudo-code") that solves the problem. **(5 points)**

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Solutions to written exam Monday 26 August

Task 1. As $\hat{\mathbf{p}} = \mathbf{p}(0)$ satisfies $\hat{\mathbf{p}}P = \hat{\mathbf{p}}$ it follows that $\hat{\mathbf{p}} = \mathbf{p}(0)$ is a stationary distribution. As $\mathbf{p}(n) = \mathbf{p}(0)P^n = (\mathbf{p}(0)P)\,P^{n-1} = \mathbf{p}(0)P^{n-1} = \ldots = \mathbf{p}(0)$ for $n \geq 0$ we have $\mathbf{E}\{X_n\} = \sum_{k=0}^{1} k\,\mathbf{P}\{X_n = k\} = \sum_{k=0}^{1} k\,\mathbf{p}(n)_k = \sum_{k=0}^{1} k\,\mathbf{p}(0)_k = \frac{\alpha}{\alpha+\beta}$ and similarly $\mathbf{E}\{X_n^2\} = \sum_{k=0}^{1} k^2\,\mathbf{p}(0)_k = \frac{\alpha}{\alpha+\beta}$ so that $\mathbf{Var}\{X_n\} = \mathbf{E}\{X_n^2\} - (\mathbf{E}\{X_n\})^2 = \ldots = \frac{\alpha\beta}{(\alpha+\beta)^2}$.

Task 2. We have $E[M(t)|F_s] = E[(X(t)-\lambda t)^2 - \lambda t|F_s] = E[(X(t)-X(s)+X(s)-\lambda t)^2 - \lambda t|F_s] = E[(X(t)-X(s))^2|F_s] + 2\,E[(X(s)-\lambda t)\,(X(t)-X(s))|F_s] + E[(X(s)-\lambda t)^2|F_s] - \lambda t = E[(X(t)-X(s))^2] + 2\,(X(s)-\lambda t)\,E[X(t)-X(s)|F_s] + (X(s)-\lambda t)^2 - \lambda t = \lambda\,(t-s) + \lambda^2(t-s)^2 + 2\,(X(s)-\lambda t)\,E[X(t)-X(s)] + (X(s)-\lambda t)^2 - \lambda t = \lambda\,(t-s) + \lambda^2(t-s)^2 + 2\,(X(s)-\lambda t)\,\lambda\,(t-s) + (X(s)-\lambda t)^2 - \lambda t = (X(s)-\lambda s)^2 - \lambda s = M(s)$ for $t \geq s$.

Task 3. $\mathbf{P}\{W(1)+W(2) > 3\} = 1 - \mathbf{P}\{\mathrm{N}(0,\sigma^2) \leq 3\} = 1 - \Phi(3/\sigma)$, where $\sigma^2 = \mathbf{E}\{(W(1)+W(2))^2\} = R_{WW}(1,1) + R_{WW}(2,2) + 2\,R_{WW}(1,2) = 2+4+4 = 10$.

Task 4. As $S_X(\omega) = 1/(1+\omega^2)$ and $S_{XY}(\omega) = 4/(4+\omega^2)$ are related by $S_{XY}(\omega) = H(\omega)\,S_X(\omega)$, we have $H(\omega) = S_{XY}(\omega)/S_X(\omega) = 4\,(1+\omega^2)/(4+\omega^2) = 4 - 3\,S_{XY}(\omega)$ so that $h(t) = 4\,\delta(t) - 3\,\mathrm{e}^{-2|t|}$.

Task 5. This is supplementary problem 9.26 in Hsu's book, which in turn is solved by insertion in known results from M/M/s queueing theory with $s = 3$, $\lambda = 30$/hour and $\mu = 20$/hour.

Task 6. See, e.g., the computational tasks of the course.