MSG800/MVE170 Basic Stochastic Processes

Written exam Tuesday 22 April 2014 2–6 am

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

GOOD LUCK!

Task 1. Consider a time homogeneous Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space E, initial distribution $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = \{0,1,2\}, \quad \mathbf{p}(0) = [1/3 \;\; 1/3 \;\; 1/3] \quad \text{and} \quad P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \end{bmatrix},$$

respectively. Write a computer program that by means of simulation finds (an approximation of) the expected vaule of the number of time units it takes until the chain has spent a total of two time units in state 0. **(5 points)**

Task 2. Find the average number of customers in an M/M/2/4 queueing system with $\lambda = \mu = 1$. **(5 points)**

Task 3. Let $\{N(t),\, t \geq 0\}$ be a unit rate (/unit intensity) Poisson process. Decide whether the process $\{\mathrm{e}^{(\ln 2)N(t)-t},\, t \geq 0\}$ is a martingale with respect to the filtration $\mathcal{F}_s$ containing information of all values $\{N(r),\, r \in [0,s]\}$ of the Poisson process up to time s. (Hint: The fact that $\mathrm{e}^{(\ln 2)N(t)-t} = 2^{N(t)}\mathrm{e}^{-t}$ can be useful.) **(5 points)**

Task 4. A WSS continuous time process $X(t)$ with autocorrelation function $R_X(\tau) = \cos(\tau)$ is input to an LTI system with a frequency response $H(\omega)$ that is symmetric [that is, $H(\omega) = H(-\omega)$]. Find the autocorrelation function $R_Y(\tau)$ of the output from the LTI system. **(5 points)**

Task 5. Find the autocorrelation function $R_X(s,t)$ of the process $X(t) = \sqrt{2}\, A\cos(U t + \Theta)$ for $t \in \mathbb{R}$, where A, U and Θ are independent random variables with A standard normal distributed, U uniformly distributed over the inteval $[0,1]$ and Θ uniformly distributed over the inteval $[0,\pi]$. (Hint: The fact that $2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$ can be useful.) **(5 points)**

Task 6. Consider a time homogeneous Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space E, initial distribution $\mathbf{p}(0)$ and transition probability matrix P given by

$$E = \{1, 2\}, \quad \mathbf{p}(0) = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix},$$

respectively, where $\alpha, \beta \in [0, 1]$ are constants such that $\alpha+\beta > 0$. Is the chain reversible?

(5 points)

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Solutions to written exam Tuesday 22 April 2014

Task 1. Here is a Mathematica program that solves the task

```
For[i=1; Result={}, i<=100000, i++,
    X = Floor[Random[UniformDistribution[{0,3}]]];
    AtZero = If[X==0, 1, 0]; Wait = 1;
    While[AtZero<2, Wait=Wait+1;
        X = {Move = Random[UniformDistribution[{0,1}]],
        If[X==0, If[Move<=1/2, Y=0, If[Move<=5/6, Y=1, Y=2]],
        If[X==1, If[Move<=2/3, Y=1, Y=2],
        If[Move<=1/2, Y=0, Y=2]]], Y}[[3]];
        If[X==0, AtZero=AtZero+1]]; AppendTo[Result,Wait]]
N[Mean[Result]]
```

Task 2. From Equation 9.36 in Hsu's book we see that $[p_0\ p_1\ p_2\ p_3\ p_4] = [\frac{8}{23}\ \frac{8}{23}\ \frac{4}{23}\ \frac{2}{23}\ \frac{1}{23}]$, so that $L = \sum_{n=0}^{4} n\, p_n = \frac{26}{23}$.

Task 3. The process is a martingale since $\mathbf{E}\{\mathrm{e}^{(\ln 2)N(t)-t}|\mathcal{F}_s\} = \mathrm{e}^{(\ln 2)N(s)-t}\,\mathbf{E}\{\mathrm{e}^{(\ln 2)(N(t)-N(s))}\} = \mathrm{e}^{(\ln 2)N(s)-t}\,\mathbf{E}\{2^{N(t)-N(s)}\} = \mathrm{e}^{(\ln 2)N(s)-t}\sum_{k=0}^{\infty} 2^k\,(t-s)^k\,\mathrm{e}^{-(t-s)}/(k!) = \mathrm{e}^{(\ln 2)N(s)-t}\mathrm{e}^{t-s} = \mathrm{e}^{(\ln 2)N(s)-s}$ for $s \leq t$.

Task 4. As $S_X(\omega) = \pi[\delta(\omega-1)+\delta(\omega+1)]$ (see Table B-2 in Hsu's book) and $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$, we get $R_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 S_X(\omega)\,\mathrm{e}^{j\omega\tau}\,d\omega = \frac{1}{2}|H(1)|^2\,\mathrm{e}^{j\tau} + \frac{1}{2}|H(-1)|^2\,\mathrm{e}^{-j\tau} = \frac{1}{2}|H(1)|^2(\mathrm{e}^{j\tau}+\mathrm{e}^{-j\tau}) = |H(1)|^2\cos(\tau)$.

Task 5. We have $R_X(s,t) = \mathbf{E}\{X(s)X(t)\} = \mathbf{E}\{A^2\cos(U(s+t)+2\Theta)\} + \mathbf{E}\{A^2\cos(U(s-t))\} = \mathbf{E}\{\cos(U(s+t)+2\Theta)\} + \mathbf{E}\{\cos(U(s-t))\} = \mathbf{E}\{\cos(U(s-t))\} = \sin(s-t)/(s-t)$.

Task 6. As $\mathbf{p}(0)\,P = \mathbf{p}(0)$ it follows that the chain is started according to its stationary distribution $\pi = \mathbf{p}(0)$. Hence the chain is reversible if and only if the detailed balance equations $\pi_i\,p_{ij} = \pi_j\,p_{ji}$ hold for $i,j = 1,2$. That the latter equations hold in turn is easy to see. And so the chain is reversible.