# MSG800/MVE170 Basic Stochastic Processes Fall 2014

# Exercise Session 5

## Chapter 6 in Hsu's book

**Solved problems.** Problems 6.14, 6.16, 6.20, 6.26, 6.27, 6.29 and 6.32 in Hsu's book.

**Supplementary problems** for own work. Problems 6.53, 6.57, 6.59, 6.62, 6.64 and 6.65 in Hsu's book.

**Computer problem** for own work. If $\{W(t)\}_{t\geq 0}$ is a Wiener process, then a so called Ornstein-Uhlenbeck (OU) process $\{X(t)\}_{t\in\mathbb{R}}$ is given by $X(t) = \mathrm{e}^{-t}\,W(\mathrm{e}^{2t})$. Using that the Wiener process is zero-mean with autocorrelation function $R_{WW}(s,t) = \min\{s,t\}$ it is not hard to establish that the OU-process is WSS zero-mean with autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}$ for $\tau \in \mathbb{R}$.

Let $\{X(t)\}_{t\in\mathbb{R}}$ and $\{O(t)\}_{t\in\mathbb{R}}$ be independent OU-process. Consider a noise process $\{N(t)\}_{t\in\mathbb{R}}$ given by $N(t) = \sqrt{2}\,\cos(\Theta + 10t)\,O(t)$, where $\Theta$ is a random variable that is independent of $X$ and $O$ and uniformly distributed over $[-\pi,\pi]$. Then $X$ has PSD $S_{XX}(\omega) = 2/(1+\omega^2)$ (see Problem 6.26) while $N$ has PSD $S_{NN}(\omega) = (S_{XX}(\omega-10) + S_{XX}(\omega+10))/2$ (see Problem 6.53): Make plots of these PSD's.

The signal $X(t)$ is sent on a noisy channel where it is disturbed by the additive noise $N(t)$ so that the recived observed signal is $Y(t) = X(t) + N(t)$. In order to reconstruct the sent signal $X(t)$ form the observed recived signal $Y(t)$ as well as possible $Y(t)$ is filtered through a so called Wiener filter with frequencey response $H(\omega) = S_{XX}(\omega)/(S_{XX}(\omega) + S_{NN}(\omega))$. The corresponding impulse response is

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{j\omega t} H(\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{\infty} \cos(\omega t)\,H(\omega)\,d\omega \approx \frac{1}{\pi}\int_{0}^{\omega_0} \cos(\omega t)\,H(\omega)\,d\omega$$

for a suitable sufficiently large $\omega_0 > 0$, say $\omega_0 = 10$ or $\omega_0 = 25$. The outsignal (attempted reconstruction of $X$) is $Z(t) = (h \star Y)(t)$.

Simulate $\{X(t)\}_{t\in[0,10]}$, $\{Y(t)\}_{t\in[0,10]}$ and $\{Z(t)\}_{t\in[0,10]}$ (with the latter computed numerically as $h\star Y$). Show by means of plots that despite the processes $X$ and $Y$ are very unlike each other, the processes $X$ and $Z$ are very alike.

In order to simulate an OU-process it is convenient to use that $X(t+\Delta) = \mathrm{e}^{-(t+\Delta)} \times (W(\mathrm{e}^{2(t+\Delta)}) - W(\mathrm{e}^{2t})) + \mathrm{e}^{-(t+\Delta)}W(\mathrm{e}^{2t}) = \mathrm{e}^{-(t+\Delta)}(W(\mathrm{e}^{2(t+\Delta)}) - W(\mathrm{e}^{2t})) + \mathrm{e}^{-\Delta}X(t)$ for $\Delta > 0$, where $\mathrm{e}^{-(t+\Delta)}(W(\mathrm{e}^{2(t+\Delta)}) - W(\mathrm{e}^{2t}))$ is $\mathrm{N}(0, 1-\mathrm{e}^{-2\Delta})$-distributed and independ-

ent of $\{X(s)\}_{s\in(-\infty,t]}$ (by defining properties of the Wiener process). Hence we may simulate, for example, a discrete sample $\{X(-15+\frac{i}{1000})\}_{i=0}^{40000}$ with step length $\Delta = \frac{1}{1000}$ of the process values $\{X(t)\}_{t\in[-15,25]}$ by taking $X(-15)$ $\mathrm{N}(0,1)$-distributed and then calculating $X(-15+\frac{i}{1000}) = \varepsilon_i + \mathrm{e}^{-1/1000}X(-15+\frac{i-1}{1000})$ recursively for $i = 1,\ldots,40000$, where $\{\varepsilon_i\}_{i=1}^{40000}$ are IID $\mathrm{N}(0, 1-\mathrm{e}^{-1/2000})$-distributed and independent of $X(-15)$.

**Note** that it is common that software give you an $\mathrm{N}(0,\sigma)$-distributed random variable instead of an $\mathrm{N}(0,\sigma^2)$-distributed one, as is e.g., the case with both Mathematica and Matlab: This is argubly the most common reason for seemingly unexplainable erroneous results from simulations. In the same manner software may give you an $\exp(1/\lambda)$-distributed random variable when you ask for an $\exp(\lambda)$-distributed one, and so on ... .