MSG800/MVE170 Basic Stochastic Processes

Written exam Thursday 4 April 2013 2–6 pm

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Good Luck!

Task 1. Let $\{X_n,\, n \geq 0\}$ be a Markov chain with transition probability matrix P. Show that $P\{X_{n+2} = j \,|\, X_n = i\} = (P^2)_{ij}$, that is, element number (i,j) of the square of P. **(5 points)**

Task 2. Let $X(t)$ be a Poisson process with rate λ. Calculate $P[X(t-d) = k \,|\, X(t) = j]$ for $0 < d < t$ and positive integers $j \geq k$. **(5 points)**

Task 3. Let $X_1, X_2, \ldots$ be independent identically distributed random variables, where each X_i can take only two values $1/2$ and 2 with the probabilities p and $1-p$, respectively. For which value of $p \in (0,1)$ is the process $\{M_n,\, n \geq 0\}$ given by $M_0 = 1$ and $M_n = \prod_{i=1}^{n} X_i = X_1\, X_2 \ldots X_n$ for $n \geq 1$ a martingale? **(5 points)**

Task 4. Let ξ and η be uncorrelated zero-mean and unit-variance random variables. Find the cross power spectral density $S_{XY}(\omega)$ of the processes $X(t)$ and $Y(t)$ given by $X(t) = \xi\, \cos(\omega_0 t) + \eta\, \sin(\omega_0 t)$ and $Y(t) = \eta\, \cos(\omega_0 t) - \xi\, \sin(\omega_0 t)$ for $t \in \mathbb{R}$, where $\omega_0 \in \mathbb{R}$ is a constant. **(5 points)**

Task 5. A wide-sense stationary continuous-time random process $X(t)$ with mean $E[X(t)] = 2$ is input to a linear time-invariant system with impulse response $h(t) = 3\,\mathrm{e}^{-2t}$ for $t \geq 0$ and $h(t) = 0$ for $t < 0$. Find the mean $E[Y(t)]$ of the output process $Y(t)$ of the system. **(5 points)**

Task 6. Describe how computer simulations can be used to generate, say 100000, observations of the waiting times $W(q)$ in the queue before being served for 100000 consecutive customers arriving to a steady-state M(1)/M(1)/2/4 queueing system. (In other words, the queueing system has exp(1)-distributed times between arrivals of new customers as well as exp(1)-distributed service times, and the queueing system has two servers and two queuing places.)

Albeit the simulation should in principle be started with the queuing system having a random number of customers according to the steady-state distribution, that is not really important in practice as it is known that the system converges very rapidly to being in steady-state. Hence one can in fact start the simulation with, e.g., 0 or 2 customers in the system. **(5 points)**

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Solutions to written exam Thursday 4 April

Task 1. We have $P\{X_{n+2} = j \,|\, X_n = i\} = P\{X_{n+2} = j,\, X_n = i\}/P\{X_n = i\} = \sum_k P\{X_{n+2} = j,\, X_{n+1} = k,\, X_n = i\}/P\{X_n = i\} = \sum_k P\{X_{n+2} = j \,|\, X_{n+1} = k,\, X_n = i\} \times P\{X_{n+1} = k,\, X_n = i\}/P\{X_n = i\} = \sum_k P\{X_{n+2} = j \,|\, X_{n+1} = k\} \times P\{X_{n+1} = k \,|\, X_n = i\} = \sum_k P_{kj} P_{ik} = (P^2)_{ij}$.

Task 2. We have $P[X(t-d) = k \,|\, X(t) = j] = P[X(t-d) = k,\, X(t) = j]/P[X(t) = j] = P[X(t-d) = k,\, X(t) - X(t-d) = j-k]/P[X(t) = j] = P[X(t-d) = k] \times P[X(t) - X(t-d) = j-k]/P[X(t) = j] = P[X(t-d) = k] \times P[X(d) = j-k]/P[X(t) = j] = \big((\lambda\,(t-d))^k \mathrm{e}^{-\lambda(t-d)}/k!\big) \times \big((\lambda\, d)^{j-k} \mathrm{e}^{-\lambda d}/(j-k)!\big) / \big((\lambda\, t)^j \mathrm{e}^{-\lambda t}/j!\big) = \ldots = \binom{j}{k}\,(1-d/t)^k (d/t)^{j-k}$.

Task 3. We have $E(M_{n+1}|F_n) = E(M_{n+1}|M_1, \ldots, M_n) = E(X_{n+1} M_n | M_n) = E(X_{n+1}) \times M_n = M_n$ when $E(X_{n+1}) = (1/2)\,p + 2\,(1-p) = 2 - 3\,p/2 = 1$, that is, when $p = 2/3$.

Task 4. As $R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \ldots = \sin(\omega_0 t)\cos(\omega_0(t+\tau)) - \cos(\omega_0 t)\sin(\omega_0(t+\tau)) = -\sin(\omega_0\tau) = \frac{1}{2j}\,(\mathrm{e}^{-j\omega_0\tau} - \mathrm{e}^{j\omega_0\tau}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\pi\,(\delta(\omega-\omega_0) - \delta(\omega+\omega_0))\,\mathrm{e}^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\,\mathrm{e}^{j\omega\tau}\,d\omega$ we have $S_{XY}(\omega) = j\pi\,(\delta(\omega-\omega_0) - \delta(\omega+\omega_0))$.

Task 5. We have $E[Y(t)] = E[\int_{-\infty}^{\infty} X(t-u)h(u)\,du] = \int_0^{\infty} E[X(t-u)]\,3\,\mathrm{e}^{-2u}\,du = 6\int_0^{\infty} \mathrm{e}^{-2u}\,du = 3$.

Task 6. You will find one solution to this problem as the array `Wait` generated by the Mathematica programme available at

`http://www.math.chalmers.se/Stat/Grundutb/GU/MSG800/A12/Exercises/Comp_Pr_6.pdf`