MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 12 January 2015 $2-6$ am

(With two figures.)

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

Motivations: All answers/solutions must be motivated. Good Luck!

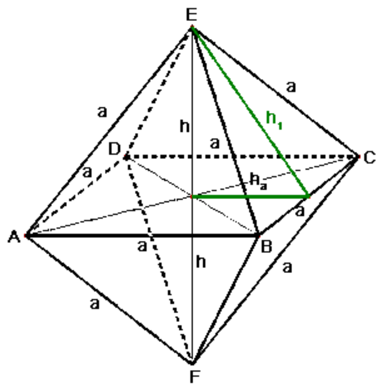
Task 1. The passport issuing service in former East Germany (=the German Democratic Republic) opened at a certain unpredictable time in the morning each day after which it was open exactly six hours after which it closed down for the day. It was forbidden for passport applicants to queue outside the passport issuing service before the opening time. When the passport issuing service opened each morning passport applicants started to arrive according to a Poisson process with arrival rate 6 applicants per hour. The passport issuing service had just one passport issuer who needed an exponential distributed time with mean 1/2 hour to issue a passport. A passport applicant that was in progress with her/his passport issuing at the closing time was abandoned (didn't get a passport). Write a computer programme that by means of stochastic simulation find an approximation of the mean number of passport applicants that got a passport each day. (Or in other words, find the expected value of the number of customers that is being finished served during the first six time units for an M/M/1 queueing system with $\lambda = 6$ and $\mu = 2$ that is started empty.)

(5 points)

Task 2. Consider a fair game runned repeatedly where the gambler looses his bet with probability 1/2 and wins double his bet with probability 1/2. The gambler starts with the bet 1 and then doubles his bet after each loss until he has his first win after which he instead bets zero forever. The gain S_n of the gambler after n bets is therefore $S_n = -(1+2+\ldots+2^{n-1}) = -(2^n-1)$ if the gambler has not had his first win yet, while the gain is instead $S_n = -(1+2+\ldots+2^{k-1}) + 2^k = 1$ if the gambler has had his first win after an earlier k'th bet were $k \leq n$. Show that S_n is a martingale. [Hint: Show that $E[S_{n+1}|S_n=1] = 1$ and $E[S_{n+1}|S_n=-(2^n-1)] = -(2^n-1)$.] **(5 points)**

Task 3. Let $\{W(t), t \geq 0\}$ be a Wiener process, that is, a zero-mean Gaussian process with autocorrelation function $R_W(s,t) = \sigma^2 \min(s,t)$. Further, let $\{N(t), t \geq 0\}$ be a Poisson process with rate (/intensity) $\lambda > 0$. Show that $R_X(s,t) = \sigma^2 \lambda \min(s,t)$ for the process $\{X(t), t \geq 0\}$ given by $X(t) = W(N(t))$. **(5 points)**

Task 4. Consider a continuous time random walk on the six corners $\{A, B, C, D, E, F\}$ of an octaeder that spends an exponentially distributed time with mean $1/4$ at each visit of an corner after which it selects one of the four neighbour corners as its next position with equal probabilities $1/4$. Show that the expected value of the time it takes the random walk to move from corner A to corner C of the octaeder (see figure below) is equal to $3/2$.



(5 points)

Task 5. Show that for a differentiable WSS continuous time random process $\{X(t), t \in \mathbb{R}\}$ with autocorrelation function $R_X(\tau)$ and derivative process $\{X'(t), t \in \mathbb{R}\}$ it holds that $E[X'(t)] = 0$ and $E[X(t)X'(t)] = R'_X(0)$. (These things are done in the book by Hsu – it does not score any points to refer to what Hsu have done, but you must either more or less redo what he does or show the asked for in some other way ... .) Also, explain why it must in fact hold that $R'_X(0) = 0$. **(5 points)**

Task 6. An LTI system with frequency response $H(\omega) = 2/(1+j\omega)$ has insignal $x(t) = \cos(t)$. Show that the outsignal is $y(t) = \cos(t) + \sin(t)$. **(5 points)**

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Solutions to written exam Monday 12 January

Task 1.

```
wiederholungen = 1000000000000000000000000000000; reisepass = 0;
for durchlauf = 1:wiederholungen
  ankunft = exprnd(1/6); betrieb = ankunft + exprnd(1/2);
  while betrieb < 6
    reisepass = reisepass + 1; ankunft = ankunft + exprnd(1/6);
    if betrieb < ankunft
      betrieb = ankunft + exprnd(1/2);
    else
      betrieb = betrieb + exprnd(1/2);
    end
  end
end
reisepass/wiederholungen
>> 11.5030473957230128452382047854
```

Alternatively, (in a fashion programmed by several students)

```
loops = 1000000000000000000000000000000; count = 0;
for i = 1:loops
  x = 1; time = exprnd(1/6);
  while time <= 6
    if x <= 1/2
      x = 1; time = time + exprnd(1/6);
    end
    time = time + exprnd(1/8); move = binornd(1,3/4);
    if move <= 1/2
      x = x - 1;
      if time <= 6
        count = count + 1;
      end
    else
      x = x + 1;
    end
  end
end
count/loops
>> 11.5030473957230128452382047854
```

Task 2. If $\{\xi_i\}_{i=1}^{\infty}$ are independent random variables with $P[\xi_i=-1]=P[\xi_i=1]=1/2$, then we have $S_1=\xi_1$ and $S_{n+1}=S_n+\xi_{n+1}(1-S_n)$ for $n\geq 1$, so that $E[S_{n+1}|F_n]=S_n+E[\xi_{n+1}]\,(1-S_n)=S_n+0\cdot(1-S_n)=S_n$, where $F_n=\sigma(S_1,\ldots,S_n)=\sigma(\xi_1,\ldots,\xi_n)$.

Alternatively, we have $E[S_{n+1}|F_n]-S_n=E[S_{n+1}-S_n|F_n]$, which in turn is $1-1=0$ for $S_n=1$ and is $(1/2)\cdot 2^n-(1/2)\cdot 2^n=0$ for $S_n=-(2^n-1)$.

Alternatively, as $E[S_{n+1}-S_n|F_n]=0$ for a fair game we get $E[S_{n+1}|F_n]=S_n$.

Alternatively, we have $E[S_{n+1}|F_n]=1=S_n$ if $S_n=1$ while $E[S_{n+1}|F_n]=-(2^{n+1}-1)\cdot(1/2)+1\cdot(1/2)=-(2^n-1)=S_n$ if $S_n=-(2^n-1)$.

Task 3. Let $\{X_n^{(\varepsilon)}\}_{n=1}^{\infty}$ be independent random variables distributed as $W(N(\varepsilon))$. As $E[W(N(\varepsilon))]=0$ and $W(N(\varepsilon))^2$ has characteristic function $\Psi^{(\varepsilon)}(\omega)=E[\mathrm{e}^{j\omega W(N(\varepsilon))^2}]=\ldots=\sum_{k=0}^{\infty}\frac{(\lambda\varepsilon)^k}{k!}\,\mathrm{e}^{-\lambda\varepsilon}/\sqrt{1-2jk\sigma^2\omega}$ we get $R_X(s,t)\sim E\big[\big(\sum_{m=0}^{[s/\varepsilon]}X_m^{(\varepsilon)}\big)\big(\sum_{n=0}^{[t/\varepsilon]}X_n^{(\varepsilon)}\big)\big]=\sum_{m=0}^{\min([s/\varepsilon],[t/\varepsilon])}E[(X_m^{(\varepsilon)})^2]=E\big[\sum_{m=0}^{\min([s/\varepsilon],[t/\varepsilon])}(X_m^{(\varepsilon)})^2\big]$, where the sum has characteristic function $(\Psi^{(\varepsilon)}(\omega))^{\min([s/\varepsilon],[t/\varepsilon])}=\big(\mathrm{e}^{-\lambda\varepsilon}(1+\lambda\varepsilon/\sqrt{1-2j\sigma^2\omega}+o(\varepsilon))\big)^{\min([s/\varepsilon],[t/\varepsilon])}\to\mathrm{e}^{-\lambda\min(s,t)(1-1/\sqrt{1-2j\sigma^2\omega})}\equiv\Psi(\omega)$ with expected value $\Psi'(0)/j=\ldots=\lambda\sigma^2\min(s,t)$ ☺.

Alternatively, we have $R_X(s,t)=E[X(s)X(t)]=E[W(N(s))W(N(t))]=E[E[W(N(s))W(N(t))]|N(s),N(t)]]=E[\sigma^2\min(N(s),N(t))]=E[\sigma^2N(s)]=\sigma^2\lambda s$ for $s\leq t$.

Alternatively, as $X(t)$ is zero-mean with independent stationary increments $R_X(s,t)=E[X(1)^2]\,s=E[W(N(1))^2]\,s=E[E[W(N(1))^2|N(1)]]\,s=E[\sigma^2N(1)]\,s=\sigma^2\lambda s$ for $s\leq t$.

Alternatively, as $X(t)$ is zero-mean with independent stationary increments we have $R_X(s,t)=E[X(t)X(s)]=E[(X(t)-X(s))X(s)]+E[X(s)^2]=E[X(s)^2]$ for $s\leq t$, where $E[X(s+\varepsilon)^2]-E[X(s)^2]=E[(X(s+\varepsilon)-X(s))^2]+2\,E[(X(s+\varepsilon)-X(s))\,X(s)]=E[(X(\varepsilon)-X(0))^2]+0=E[W(N(\varepsilon))^2]=E[W(1)^2]\,P[N(\varepsilon)=1]+\mathrm{o}(\varepsilon)=\sigma^2\lambda\,\varepsilon+\mathrm{o}(\varepsilon)$, so that $\frac{d}{ds}\,E[X(s)^2]=\sigma^2\lambda$ and $E[X(s)^2]=\sigma^2\lambda\,s$ as $E[W(N(0))^2]=0$.

Task 4. The discrete time jump processs X_n which registers all movements of $X(t)$ modified to have a mandatory jump from C to A has transition matrix

$$P=\begin{bmatrix}0&1/4&0&1/4&1/4&1/4\\1/4&0&1/4&0&1/4&1/4\\1&0&0&0&0&0\\1/4&0&1/4&0&1/4&1/4\\1/4&1/4&1/4&1/4&0&0\\1/4&1/4&1/4&1/4&0&0\end{bmatrix}$$

with stationary distribution $\pi=[\pi_{\mathrm{A}}\;\;\pi_{\mathrm{B}}\;\;\pi_{\mathrm{C}}\;\;\pi_{\mathrm{D}}\;\;\pi_{\mathrm{E}}\;\;\pi_{\mathrm{F}}]=[\frac{2}{7}\;\frac{1}{7}\;\frac{1}{7}\;\frac{1}{7}\;\frac{1}{7}\;\frac{1}{7}]$, giving $E_{\mathrm{AC}}=(\mu_{\mathrm{C}}-1)\cdot(1/4)=(1/\pi_{\mathrm{C}}-1)\cdot(1/4)=3/2$ (by starting jump process in C).

Alternatively, the (unmodified) discrete time jump processs X_n has stationary distribution $\pi = [\frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}]$ and $E_{\mathrm{AC}} = \mu_A \cdot (1/4) = (1/\pi_A) \cdot (1/4) = 3/2$. (This alternative was suggested by student "Kristoffer").

Alternativerly, writing E_{AB} and E_{AC} for the expected values of the times it takes the random walk to move from A to B and from A to C, respectively, we have

$$\begin{cases} E_{\mathrm{AB}} = 1/4 + 2 \cdot (1/4) \cdot E_{\mathrm{AB}} + (1/4) \cdot E_{\mathrm{AC}} \\ E_{\mathrm{AC}} = 1/4 + 4 \cdot (1/4) \cdot E_{\mathrm{AB}} \end{cases} \Leftrightarrow \begin{cases} E_{\mathrm{AB}} = 5/4 \\ E_{\mathrm{AC}} = 3/2 \end{cases}.$$

Alternatively, consider a birth-and-death process with values $\{A, BDEF, C\}$ and intensities $\lambda_{\mathrm{A}} = \mu_{\mathrm{C}} = 4$ and $\lambda_{\mathrm{BDEF}} = \mu_{\mathrm{BDEF}} = 2$ (as every second jump from one of B, D, E or F ends up among these), so that $E_{\mathrm{AC}} = (1/4) + (1/2) + (1/2) \cdot E_{\mathrm{AC}}$ and $E_{\mathrm{AC}} = 3/2$. In fact, $\Psi(\omega) \equiv E[\mathrm{e}^{j\omega T_{AC}}] = E[\mathrm{e}^{j\omega \exp(4)}]\, E[\mathrm{e}^{j\omega \exp(2)}]\big((1/2) + (1/2) \cdot \Psi(\omega)\big)$, so that $\Psi(\omega) = E[\mathrm{e}^{j\omega \exp(4)}]\, E[\mathrm{e}^{j\omega \exp(2)}] / \big(2 - E[\mathrm{e}^{j\omega \exp(4)}]\, E[\mathrm{e}^{j\omega \exp(2)}]\big) = \ldots = (3+\sqrt{5}\,)(3-\sqrt{5}\,)/\big((3+\sqrt{5}-j\omega)(3-\sqrt{5}-j\omega)\big)$ making T_{AC} the sum of two independent $\exp(3\pm\sqrt{5}\,)$ distributions with expected value $E[T_{\mathrm{AC}}] = 1/(3+\sqrt{5}\,) + 1/(3-\sqrt{5}\,) = \ldots = 3/2$ ☺.

Task 5. As $X'(t)$ is the output from an LTI system with input $X(t)$ and $H(\omega) = j\omega$ (see Task 6 below) and $\omega S_X(\omega)$ is odd, we have $\mu_{X'} = \mu_X H(0) = 0 = \int_{-\infty}^{\infty} j\omega\, S_X(\omega)\, d\omega$ $= \int_{-\infty}^{\infty} H(\omega) S_X(\omega)\, d\omega = R_{XX'}(0)$ where in addition $\int_{-\infty}^{\infty} j\omega\, S_X(\omega)\, d\omega = R'_X(0)$.

Alternatively, $E[\frac{d}{dt}X(t)] = \frac{d}{dt}E[X(t)] = \frac{d}{dt}\mu_X = 0 = \frac{1}{2}\frac{d}{dt}R_X(0) = \frac{1}{2}\frac{d}{dt}E[X(t)^2] = E[X(t)X'(t)] = E[X(t)\frac{d}{d\tau}X(t+\tau)]\big|_{\tau=0} = \frac{d}{d\tau}E[X(t)X(t+\tau)]\big|_{\tau=0} = R'_X(0)$.

Alternatively, as $\int_{-\infty}^{\infty} \delta'(s)x(t-s)\, ds = \int_{-\infty}^{\infty} \delta(s)x'(t-s)\, ds = x'(t)$ differentiation has impulse response $\delta'(s)$ so that $E[X'(t)] = E[(\delta' \star X)(t)] = (\delta' \star E[X])(t) = \frac{d}{dt}\mu_X = 0 = \frac{1}{2}\frac{d}{dt}R_X(0) = \frac{1}{2}(\delta' \star E[X^2])(t) = \frac{1}{2}E[(\delta' \star X^2)(t)] = E[X(t)X'(t)] =_{\text{independent of } t \text{ by previous}} E[X(0)X'(0)] = E[X(0)(\delta' \star X)(t)]\big|_{t=0} = (\delta' \star E[X(0)X(t)])\big|_{t=0} = R'_X(0)$.

Altenatively, we get $R'_X(0) = 0$ from $R_X(\tau) \leq R_X(0)$ or from that $R_X(\tau)$ is symmetric [so that $R'_X(0) = \frac{d}{d\tau}R_X(\tau)\big|_{\tau=0} = \frac{d}{d\tau}R_X(-\tau)\big|_{\tau=0} = -R'_X(0)$].

Task 6. As $(Fx')(\omega) = j\omega\, (Fx)(\omega)$ and $H(\omega) = 2/(1+j\omega)$, so that $(Fx)(\omega) = (Fy)(\omega)$ $/H(\omega) = \frac{1}{2}\big((Fy)(\omega) + (Fy')(\omega)\big)$, we have $y(t) + y'(t) = 2x(t) = 2\cos(t)$ giving $y(t)$ $= \cos(t) + \sin(t)$ (as the time-average 0 of the insignal carries over to the outsignal).

Alternatively, as $H(\omega) = 2(1-j\omega)/(1+\omega^2)$ acts as $1-j\omega$ since $x(t)$ only has unit frequencies $[(Fx)(\omega) = \pi(\delta(\omega-1)+\delta(\omega-1))]$ we have $y(t) = x(t) - x'(t) = \cos(t) + \sin(t)$.

Alternatively, $y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{j\omega t} H(\omega)\, (Fx)(\omega)\, d\omega = \int_{-\infty}^{\infty} \mathrm{e}^{j\omega t} \frac{1}{1+j\omega}\, (\delta(\omega-1)+\delta(\omega-1))$ $d\omega = \frac{1}{1+j}\,\mathrm{e}^{jt} + \frac{1}{1-j}\,\mathrm{e}^{-jt} = \frac{1-j}{2}\,\mathrm{e}^{jt} + \frac{1+j}{2}\,\mathrm{e}^{-jt} = \frac{1}{2}(\mathrm{e}^{jt} + \mathrm{e}^{-jt}) + \frac{1}{2j}(\mathrm{e}^{jt} - \mathrm{e}^{-jt}) = \cos(t) + \sin(t)$.

Alternatively, as $h(s) = 2\,\mathrm{e}^{-s}u(s)$ we have $y(t) = \int_{-\infty}^{\infty} x(t-s)\, h(s)\, ds = \int_0^{\infty} 2\cos(t-s)\,\mathrm{e}^{-s}\, ds = \int_0^{\infty} (\mathrm{e}^{j(t-s)-s} + \mathrm{e}^{-j(t-s)-s})\, ds = \frac{1}{1+j}\,\mathrm{e}^{jt} + \frac{1}{1-j}\,\mathrm{e}^{-jt} =_{\text{see above}} \cos(t) + \sin(t)$.