MSG800/MVE170 Basic Stochastic Processes

Written exam Wednesday 15 April 2015 $2-6$ pm

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

GOOD LUCK!

Task 1. Let $X(t)$ be a Poission process with arrival rate $\lambda > 0$ so that the time T to the first arrival (/to the first Poisson process count) is exponentially distributed with mean $1/\lambda$. Show that the conditional distribution $P[T \leq s \,|\, X(t) = 1]$ of T given that $X(t) = 1$ is uniform over $[0, t]$. [Hint: Note that $T \leq s$ if and only if $X(s) = 1$.] **(5 points)**

Task 2. An urn contains initially at time $n = 0$ one black and one red ball. At each time $n \geq 1$ a ball is drawn randomly from the urn and is put back in the urn together with an additional ball of the same colour. Consequently, after the n'th draw and putting back operation the urn cotains a total of $n+2$ balls. Let X_n denote the number of these $n+2$ balls that are black. Show that $M_n = X_n/(n+2)$ is a martingale. **(5 points)**

Task 3. Show that the Wiener process $X(t)$ is mean-square continuous for $t > 0$, that is, show that $E[(X(t+\varepsilon) - X(t))^2] \to 0$ as $\varepsilon \to 0$ for $t > 0$. **(5 points)**

Task 4. Let $W(t)$ be continuous time white noise, that is, a WSS zero-mean Gaussian process with autocorrelation function $R_W(\tau) = \sigma^2 \delta(\tau)$. Show that $X(t) = \int_0^t W(r)\, dr$ is a Wiener process. **(5 points)**

Task 5. Consider a M/M/1 queueing system with $\rho = \lambda/\mu < 1$. Show that the total time T a customer spends in the queueing system is exponentially distributed with mean $1/(\mu - \lambda)$. [Hint: Show that T has the moment generating function $E[\mathrm{e}^{sT}] = (\mu - \lambda)/(\mu - \lambda - s)$ of an exponential distribution with mean $1/(\mu - \lambda)$.] **(5 points)**

Task 6. Consider an M/M/1/2 queueing system with $\lambda = 1$ and $\mu = 3$. Write a program that by means of computer simulation finds an approximation of the probability $P[\max_{T \leq t \leq T+10} X(t) = 2]$ for a fixed T, that is, the probability that the queueing system gets full during a time interval of 10 time units length. (The sought after probability is not equal to $p_2 = (1-\rho)\,\rho^2/(1-\rho^3) = 1/13$, but a lot larger than that.) **(5 points)**

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Solutions to written exam Wednesday 15 April

Task 1. $P[T \le s \,|\, X(t)=1] = P[X(s)=1 \,|\, X(t)=1] = P[X(s)=1,\, X(t)=1]/P[X(t)=1] = P[X(s)=1]\, P[X(t)-X(s)=0]/P[X(t)=1] = (\lambda\, s\, \mathrm{e}^{-\lambda s})\,(\mathrm{e}^{-\lambda(t-s)})/(\lambda\, t\, \mathrm{e}^{-\lambda t}) = s/t$ for $s \in [0,t]$, so that the mentioned conditional distribution of T is uniform over $[0,t]$.

Task 2. Conditional on the value of M_n it is easy to see that $M_{n+1} = [(n{+}2)M_n+1]/(n{+}3)$ with probability M_n and $M_{n+1} = (n{+}2)M_n/(n{+}3)$ with probability $1{-}M_n$, so that $E[M_{n+1}|F_n] = M_n \times [(n{+}2)M_n{+}1]/(n{+}3) + (1{-}M_n)\times(n{+}2)M_n/(n{+}3) = \ldots = M_n$ for $F_n = \sigma(M_1,\ldots,M_m)$.

Task 3. As $R_X(s,t) = \sigma^2 \min(s,t)$ we have $E[(X(t{+}\varepsilon) - X(t))^2] = R_X(t{+}\varepsilon, t{+}\varepsilon) - 2\,R_X(t,t{+}\varepsilon) + R_X(t,t) = \sigma^2 \min(t{+}\varepsilon, t{+}\varepsilon) - 2\,\sigma^2 \min(t,t{+}\varepsilon) + \sigma^2 \min(t,t) = \sigma^2(\varepsilon - 2\min(0,\varepsilon)) = |\varepsilon| \to 0$ as $\varepsilon \to 0$ for $t > 0$.

Task 4. As $X(t)$ inherits the property of $W(t)$ to be a zero-mean Gaussian process it is enough to check that $R_X(s,t) = \sigma^2 \min(s,t)$. But $R_X(s,t) = E[(\int_0^s W(q)\,dq)\,(\int_0^t W(r)\,dr)] = \int_0^s \int_0^t E[W(q)W(r)]\,dqdr = \int_0^{\min(s,t)} \int_0^{\min(s,t)} \sigma^2 \delta(r-q)\,dqdr = \int_0^{\min(s,t)} \sigma^2\,dr = \sigma^2 \min(s,t)$.

Task 5. As T is the sum of the service time of the customer under consideration plus the service times of the $X(t)$ customers before that customer queueing for service, it follows that T is the sum of $X(t)+1$ independent exponentially distributed random variables with mean $1/\mu$. As $X(t)$ has the stationary distribution $P[X(t)=n] = (1-\lambda/\mu)\,(\lambda/\mu)^n$ for $n \ge 0$ it follows that $E[\mathrm{e}^{sT}] = \sum_{n=0}^{\infty} E[\mathrm{e}^{sT}|X(t)=n]\,P[X(t)=n] = \sum_{n=0}^{\infty} E[\mathrm{e}^{s(T_1+\ldots+T_{n+1})}]\,(1-\lambda/\mu)\,(\lambda/\mu)^n = \sum_{n=0}^{\infty}(E[\mathrm{e}^{sT_1}])^{n+1}\,(1-\lambda/\mu)\,(\lambda/\mu)^n = \sum_{n=0}^{\infty}(\mu/(\mu-s))^{n+1}\,(1-\lambda/\mu)\,(\lambda/\mu)^n = \ldots = (\mu-\lambda)/(\mu-\lambda-s)$, were $T_1, T_2, \ldots$ are independent exponentially distributed random variables with mean $1/\mu$.

Task 6.

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In[1]:= {lam,mu} = {1,3}; {Max2,Rep} = {0,100000}; {rho,p0,p1,p2} =
          {lam/mu, (1-rho)/(1-rho^3), rho*p0, rho*p1};
In[2]:= For[i=1, i<=Rep, i++, {Time,X} = {0, {xi=Random[], If[xi<=p0, 0,
          If[xi<=p0+p1,1,2]]}[[2]]}; While[Time<10 && X<2, {Arr,Ser} =
           {Random[ExponentialDistribution[lam]], Random[ExponentialDistribution[mu]]};
           If[X==0, Time=Time+Arr; X=1, Time=Time+Min[Arr,Ser]; If[Arr<Ser,
           X=X+1, X=X-1]]]; If[Time<10, Max2=Max2+1]];
In[3]:= N[Max2/Rep]
Out[2]= 0.88745
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