MSG800/MVE170 Basic Stochastic Processes

Exercise Session 6

Archetypical type-problems of typical type-exam-type for own work

Archetypical type-problem of typical type-exam-type 1. Consider a Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space E and transition matrix P given by

$$E = \{0,1,2\} \qquad \text{and} \qquad P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix},$$

respectively, and let $E_i = \mathbf{E}\big\{\min\{n \geq 1 : X_n = i\} \,\big|\, X_0 = i\big\}$ for $i = 0,1,2$. Show that the chain has stationary distribution $[1/E_0 \;\; 1/E_1 \;\; 1/E_2]$.

Archetypical type-problem of typical type-exam-type 2. Let $\{W(t)\}_{t\geq 0}$ be a Wiener process and $\lambda > 0$ a constant. Show that $\{W(\lambda t)\}_{t\geq 0}$ is also a Wiener process.

Archetypical type-problem of typical type-exam-type 3. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. Show that $\{\mathrm{e}^{-\lambda t} 2^{N(t)}\}_{t\geq 0}$ is a martingale wrt. the filtration F_s, $s \geq 0$, containing all information about the process values $\{N(r)\}_{r\leq s}$.

Archetypical type-problem of typical type-exam-type 4. Let $\{e_n\}_{n\in\mathbb{Z}}$ be uncorrelated zero-mean and unit variance random variables (i.e., discrete time white noise). Find the autocorrelation function of the process $\{X_n\}_{n\in\mathbb{Z}}$ given by $X_n = e_n + e_{n-1}/2$.

Archetypical type-problem of typical type-exam-type 5. Recall that differentiating a random process $\{X(t)\}_{t\in\mathbb{R}}$ corresponds to processing the process through a linear system with frequency responce $H(\omega) = j\omega$. Show that the derivative of a WSS process X with autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}$ has autocorrelation function $R_{X'X'}(\tau) = 2\delta(\tau) - \mathrm{e}^{-|\tau|}$.

Archetypical type-problem of typical type-exam-type 6. Let $N(t)$ for $t \geq 0$ denote the total number of customers in a M/M/2/4 queuing system with exp(1)-distributed times between arriving customers as well as exp(1)-distributed service times. Assume that $N(0) = 0$ and let $\{T_n\}_{n=1}^{\infty}$ be the strictly increasing sequence of random times at which $\{N(t)\}_{t\geq 0}$ changes its values, that is, $T_{n+1} = \min\{t > T_n : N(t) \neq N(T_n)\}$ for $n \in \mathbb{N}$, with the convention $T_0 = 0$. Find the transition matrix P for the Markov chain $\{X_n\}_{n=0}^{\infty}$ with state space $E = \{0,1,2,3,4\}$ given by $X_n = N(T_n)$.