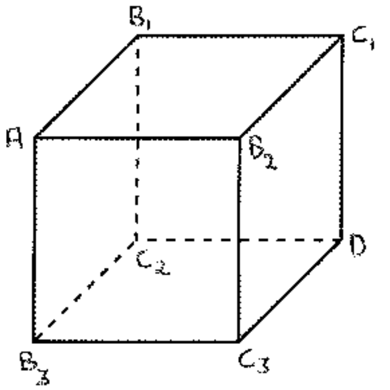
Exercise 6.3.4 in Grimmett and Stirzaker

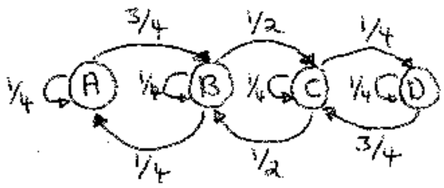
Task. A particle performs a discrete time random walk on the vertices of a cube. At each step it remains where it is with probability 1/4 or moves to one of its neighbouring vertices each having probability 1/4. Let A and D denote two diametrically opposite vertices. If the walk starts at A, find

(a) the mean number of steps until its first visit to D,

(b) the mean number of steps until its first return to A, and

(c) the mean number of visits to D before its first return to A.





Solution. (a) Let B_1, B_2 and B_3 denote the three vertices that are closest to A (= one step away from A) and C_1, C_2 and C_3 the three vertices that are closest to D (= one step away from D = two steps away from A). Introduce a four state Markov chain with values A, B, C and D indicating if the random walk is in A, in one of the states B_1, B_2 or B_3, in one of the states C_1, C_2 and C_3, or in the state D, respectively, with corresponding probability transition matrix

$$P = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 3/4 & 1/4 \end{bmatrix}.$$

Writing T_{AD}, T_{BD} and T_{CD} for the mean number of steps until the first visit to D starting at A, B and C, respectively, we then have the following system of equations

$$\begin{cases} \mathbf{E}\{T_{\mathrm{AD}}\} = 1 + (1/4)\cdot\mathbf{E}\{T_{\mathrm{AD}}\} + (3/4)\cdot\mathbf{E}\{T_{\mathrm{BD}}\} \\ \mathbf{E}\{T_{\mathrm{BD}}\} = 1 + (1/4)\cdot\mathbf{E}\{T_{\mathrm{AD}}\} + (1/4)\cdot\mathbf{E}\{T_{\mathrm{BD}}\} + (1/2)\cdot\mathbf{E}\{T_{\mathrm{CD}}\} \\ \mathbf{E}\{T_{\mathrm{CD}}\} = 1 + (1/2)\cdot\mathbf{E}\{T_{\mathrm{BD}}\} + (1/4)\cdot\mathbf{E}\{T_{\mathrm{CD}}\} + (1/4)\cdot 0 \end{cases},$$

which in turn is obtained by conditioning on where we end up after one step on our journey to D starting at A, B and C, respectively. Solving this

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In[1]:= Solve[{AD, BD, CD} == {1+AD/4+3*BD/4,
                  1+AD/4+BD/4+CD/2, 1+BD/2+CD/4}, {AD, BD, CD}]
Out[1]= {AD -> 40/3, BD -> 12, CD -> 28/3}
```

we arrive at the answer $\mathbf{E}\{T_{\mathrm{AD}}\} = 40/3$.

(b) Writing T_{AA} and T_{BA} for the mean number of steps until the next visit to A starting at A and B, respectively, we may use the result of task (a) together with some obvious symmetry properties to obtain

$$\mathbf{E}\{T_{\mathrm{AA}}\} = 1 + (1/4)\cdot 0 + (3/4)\cdot \mathbf{E}\{T_{\mathrm{BA}}\} = 1 + (3/4)\cdot \mathbf{E}\{T_{\mathrm{CD}}\} = 1 + 28/4 = 8.$$

(c) Write D_{AA}, D_{BA}, D_{CA} and D_{DA} for the mean number of visits to D before next visit to A when starting at A, B, C and D, respectively. In the fashion of the solution to task (a) we then have

$$\begin{cases} \mathbf{E}\{D_{\mathrm{AA}}\} = (1/4)\cdot 0 + (3/4)\cdot \mathbf{E}\{D_{\mathrm{BA}}\} \\ \mathbf{E}\{D_{\mathrm{BA}}\} = (1/4)\cdot 0 + (1/4)\cdot \mathbf{E}\{D_{\mathrm{BA}}\} + (1/2)\cdot \mathbf{E}\{D_{\mathrm{CA}}\} \\ \mathbf{E}\{D_{\mathrm{CA}}\} = (1/2)\cdot \mathbf{E}\{D_{\mathrm{BA}}\} + (1/4)\cdot \mathbf{E}\{D_{\mathrm{CA}}\} + (1/4)\cdot (\mathbf{E}\{D_{\mathrm{DA}}\}+1) \\ \mathbf{E}\{D_{\mathrm{DA}}\} = (3/4)\cdot \mathbf{E}\{D_{\mathrm{CA}}\} + (1/4)\cdot (\mathbf{E}\{D_{\mathrm{DA}}\}+1) \end{cases}$$

(remember that we now are counting the number of visits to D, not time, so we should not add time units on the right-hand side, but instead possible visits to D), with solution

```
In[2]:= Solve[{AA, BA, CA, DA} == {3*BA/4, BA/4+CA/2,
                 BA/2+CA/4+(DA+1)/4, 3*CA/4+(DA+1)/4}, {AA, BA, CA, DA}]
Out[2]= {AA -> 1, BA -> 4/3, CA -> 2, DA -> 7/3}
```

giving us the answer $\mathbf{E}\{D_{\mathrm{AA}}\} = 1$.

Exercise 6.9.10 in Grimmett and Stirzaker

Task. Let $X(t)$ be a birth-death process with $\lambda_n = \lambda > 0$ and $\mu_n = \mu > 0$, where $\lambda > \mu$ and $X(0) = 0$. [Or in other words, $X(t)$ is the number of customers in an M/M/1 queueing system starting up empty and with traffic intensity $\rho > 1$.] Show that the total time T_i spent in state i is $\exp(\lambda - \mu)$-distributed.

Solution. Writing q_i for the probability of ever visiting 0 having started at i we have

$$q_0 = 1 \qquad \text{and} \qquad q_i = \frac{\mu}{\lambda+\mu}\, q_{i-1} + \frac{\lambda}{\lambda+\mu}\, q_{i+1} \quad \text{for } i \geq 1.$$

The zeros of the characteristic polynomial for this difference equation are

$$p(x) = \frac{\lambda}{\lambda+\mu}\, x^2 - x + \frac{\mu}{\lambda+\mu} = 0 \;\;\Leftrightarrow\;\; x = \mu/\lambda \;\text{ or }\; x = 1,$$

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In[3]:= Solve[lambda*x^2/(lambda+mu)-x+mu/(lambda+mu) == 0, {x}]
Out[3]= {{x -> mu/lambda}, {x -> 1}}
```

so that $q_i = A\,(\mu/\lambda)^i + B\,1^i$ for some constants $A, B \in \mathbb{R}$. As we must have $q_i \to 0$ as $i \to \infty$ we have $B = 0$ after which $q_0 = 1$ gives $A = 1$, so that $q_i = (\mu/\lambda)^i$ for $i \geq 0$

To find T_0 we note that this time is the sum of the $\exp(\lambda)$-distributed time it takes to leave 0 plus another independent $\exp(\lambda)$-distributed time added for each revisit of 0, where the number N of such revisits has PMF $\mathbf{P}\{N = n\} = (\mu/\lambda)^n(1-\mu/\lambda)$ for $n \geq 0$. As the CHF of an $\exp(\lambda)$-distributed random variable is $\mathbf{E}\{e^{j\omega \exp(\lambda)}\} = \lambda/(\lambda - j\omega)$

```
In[4]:= Integrate[Exp[I*omega*x]*lambda*Exp[-lambda*x],
              {x,0,Infinity}, Assumptions->lambda>0&&omega∈Reals]
Out[4]= lambda/(lambda - I omega)
```

it follows that (making use of the basic fact that the CHF of a sum of independent random variables is the product of the CHF's of the individual random variables)

$$\mathbf{E}\{e^{j\omega T_0}\} = \frac{\lambda}{\lambda - j\omega} \times \sum_{n=0}^{\infty}\Big(\frac{\lambda}{\lambda - j\omega}\Big)^n (\mu/\lambda)^n(1-\mu/\lambda) = \frac{\lambda-\mu}{\lambda-\mu-j\omega},$$

```
In[5]:= lambda/(lambda-I*omega)*Sum[(lambda/(lambda-I*omega))^n*
                      (mu/lambda)^n*(1-mu/lambda), {n,0,Infinity}]
Out[5]= (lambda - mu)/(lambda - mu - I omega)
```

To find T_i we note that (by considering what the first state after having left i is – $i-1$ or $i+1$) the probability of ever returning to i having started there is

$$\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot q_1 = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot \frac{\mu}{\lambda} = \frac{2\,\mu}{\lambda+\mu}.$$

As the time spent at each visits of i is $\exp(\lambda+\mu)$-distributed it follows as above that

$$\mathbf{E}\{e^{j\omega T_i}\} = \frac{\lambda+\mu}{\lambda+\mu-j\omega} \times \sum_{n=0}^{\infty}\Big(\frac{\lambda+\mu}{\lambda+\mu-j\omega}\Big)^n\Big(\frac{2\,\mu}{\lambda+\mu}\Big)^n\Big(1-\frac{2\,\mu}{\lambda+\mu}\Big) = \frac{\lambda-\mu}{\lambda-\mu-j\omega},$$

```
In[6]:= (lambda+mu)/(lambda+mu-I*omega)*
                         Sum[(2*mu/(lambda+mu-I*omega))^n*
                                       (1-2*mu/(lambda+mu)), {n,0,Infinity}]
Out[6]= (lambda - mu)/(lambda - mu - I omega)
```

Exercise 6.11.4 in Grimmett and Stirzaker

Task. Let $X(t)$ be a birth-death process with $\lambda_n = n\lambda$ and $\mu_n = n\mu$, where $0 < \lambda < \mu$ and $X(0) = 1$. Show that the distribution of $X(t)$ conditional on the event that $\{X(t) > 0\}$ converges as $t \to \infty$ to a geometric distribution.

Solution. By Theorem 6.11.10 in G&S $X(t)$ has probability generating function

$$G(s,t) = \sum_{n=0}^{\infty} s^n \,\mathbf{P}\{X(t) = n\} = \frac{\mu\,(1-s) - (\mu - \lambda s)\,\mathrm{e}^{-t(\lambda-\mu)}}{\lambda\,(1-s) - (\mu - \lambda s)\,\mathrm{e}^{-t(\lambda-\mu)}}.$$

The probability generating function of $X(t)$ conditional on that $\{X(t) > 0\}$ is therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} s^n \,\mathbf{P}\{X(t) = n \,|\, X(t) > 0\} &= \sum_{n=1}^{\infty} s^n \,\frac{\mathbf{P}\{X(t) = n\}}{\mathbf{P}\{X(t) > 0\}} \\
&= \frac{G(s,t) - \mathbf{P}\{X(t) = 0\}}{\mathbf{P}\{X(t) > 0\}} \\
&= \frac{G(s,t) - G(0,t)}{1 - G(0,t)} \\
&= \frac{(\mu - \lambda)\,s\,\mathrm{e}^{-t(\lambda-\mu)}}{(\mu - \lambda s)\,\mathrm{e}^{-t(\lambda-\mu)} - \lambda(1-s)} \\
&\to \frac{(\mu - \lambda)\,s}{\mu - \lambda s} \quad \text{as } \ t \to \infty \\
&= \frac{(1-\rho)\,s}{1 - \rho s} \quad \text{where } \ \rho = \lambda/\mu \\
&= (1-\rho)\,s \sum_{n=0}^{\infty} (\rho s)^n \\
&= \sum_{n=1}^{\infty} s^n \,\rho^{n-1}(1-\rho)
\end{aligned}$$

```
In[7]:= Clear[G]; G[s_,x_] := (mu*(1-s)-(mu-lambda*s)*x)/
                                       (lambda*(1-s)-(mu-lambda*s)*x)
In[8]:= Simplify[(G[s,x]-G[0,x])/(1-G[0,x])]
Out[8]= ((mu-lambda)*s*x) / ((mu-lambda*s)*x-lambda*(1-s))
```

so that $\mathbf{P}\{X(t) = n \,|\, X(t) > 0\} = \rho^{n-1}(1 - \rho)$ for $n \geq 1$ indeed is geometrically distributed.

Solution to computational problem

```
clc, clf
   format long
   test = 1000000;
   ok = 0;
   for i = 1:test
     t = 0;
     q = rand();
     x = -1;
     a = 0;
     for j = 0:9
       a = a + 1/2^(j+1);
       if q < a
         x = j;
         break
       end
     end
     if x <= -1/2
       x = 10;
     end
     while t <= 10
       if x >= 9+1/2
         ok = ok + 1;
         break
       end
       birth = exprnd(1);
       death = exprnd(1/2);
       if birth < death || x <= 1/2
         t = t + birth;
         x = x + 1;
       else
         t = t + death;
         x = x-1;
       end
     end
   end
   ok/test
```