MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 16 December 2013 2–6 am

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated.

GOOD LUCK!

Task 1. Consider the discrete time random process $\{X_n,\, n \geq 1\}$ given by $X_n = \cos(nU)$ for $n \geq 1$, where U is a random variable that is uniformly distributed over the interval $[-\pi, \pi]$. Show that the process $\{X_n,\, n \geq 1\}$ is wide-sense stationary. (Hint: The formula $\cos(x)\cos(y) = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(y-x)$ can be useful.) **(5 points)**

Task 2. The one and same zero-mean wide sense stationary random process $\{X(t),\, t \in \mathbb{R}\}$ with autocorrelation function $R_X(s,t) = \mathrm{e}^{-3|t-s|}$ is input to two different continuous-time linear time-invariant system with outputs $\{Y_1(t),\, t \in \mathbb{R}\}$ and $\{Y_2(t),\, t \in \mathbb{R}\}$, respectively, and impulse responses $h_1(t) = \mathrm{e}^{-t}$ for $t \geq 0$, $h_1(t) = 0$ for $t < 0$ and $h_2(t) = \mathrm{e}^{-2t}$ for $t \geq 0$, $h_2(t) = 0$ for $t < 0$, respectively. Find $E(Y_1(t)Y_2(t))$. **(5 points)**

Task 3. Let $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2}\, dy$ and $\phi(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$ be the standard normal (zero-mean and unit-variance) cummulative probability distribution function and the standard normal probability density function, respectively. Find a random process $\{X(t),\, t \in \mathbb{R}\}$ such that the following calculation is valid:

$$\begin{aligned}
P(X(1) \leq 0,\, X(2) \leq 0) &= P\big(X(1) \leq 0,\, X(2) - \tfrac{1}{\sqrt{2}}\, X(1) \leq -\tfrac{1}{\sqrt{2}}\, X(1)\big) \\
&= \int_{-\infty}^{0} P\big(X(2) - \tfrac{1}{\sqrt{2}}\, X(1) \leq -\tfrac{1}{\sqrt{2}}\, x\big)\, \phi(x)\, dx \\
&= \int_{-\infty}^{0} \Phi(-x)\, \phi(x)\, dx \\
&= \int_{0}^{\infty} \Phi(x)\, \phi(x)\, dx \\
&= \left[\frac{\Phi(x)^2}{2}\right]_0^{\infty} \\
&= \frac{3}{8}
\end{aligned}$$

(5 points)

Continuation on next page!

Task 4. Let $\{W(t),\, t \geq 0\}$ be a Wiener process with $E(W(1)^2) = 1$. Show that the random process $\{\int_0^t W(u)\, du - \frac{1}{3}W(t)^3,\, t \geq 0\}$ is a martingale with respect to the filtration $\mathcal{F}_s$ containing information of all values $\{W(u),\ u \in [0,s]\}$ of the Wiener process up to time s. **(5 points)**

Task 5. Calculate the limit $\lim_{s,t\to\infty} R_X(s, s+t) = \lim_{s,t\to\infty} E(X(s)X(s+t))$ for a continuous time Markov chain $\{X(t)\,;\, t \geq 0\}$ with state space (possible values) S and generator G given by

$$S = \{0,1\} \qquad \text{and} \qquad G = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix},$$

respectively, where $\alpha, \beta > 0$ are given constants. **(5 points)**

Task 6. Let $X(t)$ denote the total number of customers at time $t \geq 0$ in an M/M/2/4 queuing system in steady-state (/started according to its stationary distribution) with Poisson arrival process with rate $\lambda = 1$ and with exponentially distributed service times with mean 1. Describe a numerical procedure (/simulation procedure) that gives an approximative numerical value for the probability $P\big(\max_{0\leq t\leq 10} X(t) = 4\big)$ that the queuing system gets full during the first 10 time units. **(5 points)**

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Solutions to written exam Monday 16 December

Task 1. Clearly $E(X_n)=0$ and $E(X_nX_{n+m})=\frac{1}{2}E(\cos((2n+m)U))+\frac{1}{2}E(\cos(mU))=0+\frac{1}{2}\delta(m)$ do not depend on n so that X_n is wide sense stationary.

Task 2. We have $E(Y_1(t)Y_2(t))=E\big(\int_{-\infty}^{\infty}h_1(u)X(t-u)\,du\int_{-\infty}^{\infty}h_2(v)X(t-v)\,dv\big)=E\big(\int_0^{\infty}\mathrm{e}^{-u}X(t-u)\,du\int_0^{\infty}\mathrm{e}^{-2v}X(t-v)\,dv\big)=\int_0^{\infty}\int_0^{\infty}\mathrm{e}^{-u}\mathrm{e}^{-2v}E(X(t-u)X(t-v))\,dudv=\int_0^{\infty}\int_0^{\infty}\mathrm{e}^{-u}\mathrm{e}^{-2v}\mathrm{e}^{-3|u-v|}\,dudv=\int_{u=0}^{u=\infty}\int_{v=u}^{v=\infty}\mathrm{e}^{-u}\mathrm{e}^{-2v}\mathrm{e}^{-3(v-u)}\,dvdu+\int_{v=0}^{v=\infty}\int_{u=v}^{u=\infty}\mathrm{e}^{-u}\mathrm{e}^{-2v}\mathrm{e}^{-3(u-v)}\,dudv=\int_{u=0}^{u=\infty}\mathrm{e}^{2u}\int_{v=u}^{v=\infty}\mathrm{e}^{-5v}\,dvdu+\int_{v=0}^{v=\infty}\mathrm{e}^{v}\int_{u=v}^{u=\infty}\mathrm{e}^{-4u}\,dudv=\int_{u=0}^{u=\infty}\frac{1}{5}\mathrm{e}^{-3u}\,du+\int_{v=0}^{v=\infty}\frac{1}{4}\mathrm{e}^{-3v}\,dv=\frac{1}{15}+\frac{1}{12}=\frac{3}{20}$.

Task 3. A zero-mean unit-variance Gaussian process with $E(X(1)X(2))=\frac{1}{\sqrt{2}}$, because for such a process $X(1)$ and $X(2)-\frac{1}{\sqrt{2}}X(1)$ are independent and $P\big(X(2)-\frac{1}{\sqrt{2}}X(1)\leq-\frac{1}{\sqrt{2}}x\big)=\Phi(-x)$ since $X(2)-\frac{1}{\sqrt{2}}X(1)$ is a Gaussian random variable with variance $\frac{1}{2}$.

Task 4. We have

$$\begin{aligned}
&E\big(\textstyle\int_0^t W(u)\,du-\frac{1}{3}W(t)^3\big|\mathcal{F}_s\big)\\
&\quad=E\big(\textstyle\int_s^t(W(u)-W(s))\,du\big|\mathcal{F}_s\big)+E\big((t-s)\,W(s)\big|\mathcal{F}_s\big)+E\big(\int_0^s W(u)\,du\big|\mathcal{F}_s\big)\\
&\qquad-\tfrac{1}{3}E\big((W(t)-W(s))^3\big|\mathcal{F}_s\big)-E\big((W(t)-W(s))^2W(s)\big|\mathcal{F}_s\big)\\
&\qquad-E\big((W(t)-W(s))\,W(s)^2\big|\mathcal{F}_s\big)-\tfrac{1}{3}E\big(W(s)^3\big|\mathcal{F}_s\big)\\
&\quad=E\big(\textstyle\int_s^t(W(u)-W(s))\,du\big)+(t-s)\,W(s)+\int_0^s W(u)\,du\\
&\qquad-\tfrac{1}{3}E\big((W(t)-W(s))^3\big)-E\big((W(t)-W(s))^2\big)\,W(s)\\
&\qquad-E\big(W(t)-W(s)\big)\,W(s)^2-\tfrac{1}{3}W(s)^3\\
&\quad=0+(t-s)\,W(s)+\textstyle\int_0^s W(u)\,du-0-(t-s)W(s)-0-\frac{1}{3}W(s)^3\\
&\quad=\textstyle\int_0^s W(u)\,du-\frac{1}{3}W(s)^3\quad\text{for }\ 0\leq s\leq t.
\end{aligned}$$

Task 5. The chain is irreducible with stationary distribution $\pi=(\frac{\beta}{\alpha+\beta}\ \ \frac{\alpha}{\alpha+\beta})$ (as this gives $\pi G=0$). Noting that $X(s)X(t)=1$ when both $X(s)$ and $X(t)$ are 1 while $X(s)X(t)=0$ otherwise it follows that $E(X(s)X(t))=(\mu^{(s)})_1\,p_{11}(t)=(\mu^{(0)}P_s)_1\,p_{11}(t)=\big((\mu^{(0)})_0\,p_{01}(s)+(\mu^{(0)})_1\,p_{11}(s)\big)\,p_{11}(t)\to\big((\mu^{(0)})_0\,\pi_1+(\mu^{(0)})_1\,\pi_1\big)\,\pi_1=\pi_1^2=\alpha^2/(\alpha+\beta)^2$ as $s,t\to\infty$.

Task 6.

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Clear[slump, start, xt, tid];

In[3]:= N[Mean[Table[{tid = 0, xt = {slump = Random[UniformDistribution[{0, 1}]], If[slump < 8/23, start = 0],
            If[8/23 ≤ slump < 16/23, start = 1], If[16/23 ≤ slump < 20/23, start = 2],
            If[20/23 ≤ slump < 22/23, start = 3], If[22/23 ≤ slump, start = 4], start}[[7]],
        While[tid < 10 && xt < 7/2, If[xt < 1/2, tid = tid + Random[ExponentialDistribution[1]]; xt = 1,
          If[1/2 < xt < 3/2,
            tid = tid + Random[ExponentialDistribution[2]]; slump = Random[UniformDistribution[{0, 1}]];
            If[slump ≤ 1/2, xt = xt + 1, xt = xt - 1], tid = tid + Random[ExponentialDistribution[3]];
            slump = Random[UniformDistribution[{0, 1}]]; If[slump ≤ 1/3, xt = xt + 1, xt = xt - 1]]]],
        If[xt > 7/2 && tid < 10, 1, 0]}[[4]], {i, 1, 100000}]]]

Out[3]= 0.46679
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