MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 25 August 2014 8.30–12.30 pm

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Calculate the probability $P\{X(1)=1,\, X(3)=3\,|\,X(2)=2\}$ for a Poisson process $X(t)$ with rate $\lambda=1$. **(5 points)**

Task 2. A discrete time process $\{X_n,\, n\geq 0\}$ is given recursively by taking X_0 zero-mean normal distributed with variance 4/3 and $X_{n+1}=X_n/2+e_n$ for $n=0,1,2,\ldots$, where $\{e_n,\, n\geq 0\}$ are independent zero-mean normal distributed random variables with variance 1 that are independent of X_0. Write a programme that by means of simulation finds an approximation of the probability $P\{\max_{0\leq n\leq 10} X_n\geq 3\}$. **(5 points)**

Task 3. Consider a birth-death process $\{X(t) : t\geq 0\}$ with $X(0)=0$ and with unit birth and death rates $\lambda_0=\lambda_1=\ldots=\mu_1=\mu_2=\ldots=1$. Find the expected value $E[T]$ of the time $T=\min\{t\geq 0 : X(t)=2\}$ it takes $X(t)$ to reach the state 2. **(5 points)**

Task 4. Let $\{X(t),\, t\geq 0\}$ be a standard Wiener process (so that $\mathrm{Var}[X(t)]=t$). Show that the process $\{\mathrm{e}^{X(t)-t/2},\, t\geq 0\}$ is a maringale with respect to the filtration F_s containing information about all process values $\{X(r)\}_{r\in[0,s]}$ of the Wiener process up to time s. (**Hint:** You may want to make use of the fact that $E[\mathrm{e}^Y]=\mathrm{e}^{s^2/2}$ for Y a zero-mean normal distributed random variable with variance s^2.) **(5 points)**

Task 5. A WSS discrete-time process $X(n)$ with power spectral density $S_X(\Omega)$ is input to a discrete-time LTI system with frequency response $H(\Omega)$ and output $Y(n)$. Show that the cross power spectral density between $X(n)$ and $Y(n)$ is given by $S_{XY}(\Omega)=H(\Omega)S_X(\Omega)$. **(5 points)**

Task 6. Consider a Markov chain with state space S and transition matrix P given by

$$S=\{0,1\}\qquad\text{and}\qquad P=\begin{bmatrix}1-\alpha & \alpha\\ \beta & 1-\beta\end{bmatrix},$$

respectively, where $\alpha,\beta\in[0,1]$ are constants. Under what additional constraints on α and β (other than that they lie in the interval $[0,1]$) does the chain possess a uniquely determined (that is, one and only one) stationary distribution π? **(5 points)**

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Solutions to written exam Monday 25 August 2014

Task 1. $P\{X(1)=1,\,X(3)=3|X(2)=2\} = P\{X(1)=1,X(2)=2,\,X(3)=3\}/P\{X(2)=2\} = P\{X(1)=1,X(2)-X(1)=1,\,X(3)-X(2)=1\}/P\{X(2)=2\} = P\{X(1)=1\}\,P\{X(2)-X(1)=1\}\,P\{X(3)-X(2)=1\}/P\{X(2)=2\} = (P\{X(1)=1\})^3/P\{X(2)=2\} = (\mathrm{e}^{-\lambda\cdot 1}(\lambda\cdot 1)^1/(1!))^3/(\mathrm{e}^{-\lambda\cdot 2}(\lambda\cdot 2)^2/(2!)) = 1/(2\,\mathrm{e})$.

Task 2.

```
In[1]:= For[j=1; Rep=100000; Big3=0, j<=Rep, j++,
          For[i=1; X0=Random[NormalDistribution[0,Sqrt[4/3]]]; Xmax=X0,
            i<=10, i++, X=X0/2+Random[NormalDistribution[0,1]];
            If[X>Xmax, Xmax=X]; X0=X];
          If[Xmax>=3, Big3=Big3+1]];
        N[Big3/Rep]
Out[1]= 0.04538
```

Task 3. It takes on the average 1 time unit (the mean of an exponential random variable with parameter λ_0) to reach the state 1. Then the chain spend on the average 1/2 time unit (the mean of an exponential random variable with parameter $\lambda_1+\mu_1$) in that state until it with equal probabilities 1/2 either jumps back to state zero or jumps up to state 2. This gives the equation $E[T]=1+1/2+(1/2)\,E[T]$ with solution $E[T]=3$.

Task 4. As $X(t)-X(s)$ is $\mathrm{N}(0,t-s)$ distributed and independent of F_s we use the hint to obtain $E(\mathrm{e}^{X(t)-t/2}|\,F_s) = E(\mathrm{e}^{X(t)-X(s)}\mathrm{e}^{X(s)-t/2}|\,F_s) = E(\mathrm{e}^{X(t)-X(s)}|\,F_s)\,\mathrm{e}^{X(s)-t/2} = E[\mathrm{e}^{X(t)-X(s)}]\,\mathrm{e}^{X(s)-t/2} = \mathrm{e}^{(t-s)/2}\,\mathrm{e}^{X(s)-t/2} = \mathrm{e}^{X(s)-s/2}$ for $t\geq s$.

Task 5. We have $S_{XY}(\Omega)=\sum_{\tau=-\infty}^{\infty}\mathrm{e}^{-j\Omega\tau}R_{XY}(\tau)=\sum_{\tau=-\infty}^{\infty}\mathrm{e}^{-j\Omega\tau}E[X(n)Y(n{+}\tau)] = \sum_{\tau=-\infty}^{\infty}\mathrm{e}^{-j\Omega\tau}E\big[X(n)\sum_{k=-\infty}^{\infty}h(k)X(n+\tau-k)\big] = \sum_{\tau=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}h(k)\,\mathrm{e}^{-j\Omega\tau}E[X(n)\,X(n+\tau-k)] = \sum_{\tau=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}h(k)\,\mathrm{e}^{-j\Omega\tau}R_X(\tau-k) = \sum_{k=-\infty}^{\infty}\mathrm{e}^{-j\Omega k}\,h(k)\big(\sum_{\tau=-\infty}^{\infty}\mathrm{e}^{-j\Omega(\tau-k)}R_X(\tau{-}k)\big) = H(\Omega)S_X(\Omega)$.

Task 6. By basic algebraic manipulations we see that

$$\left\{\begin{array}{c}\pi P=\pi\\ \pi_0{+}\pi_1=1\end{array}\right. \Leftrightarrow \left\{\begin{array}{c}\pi_0(1{-}\alpha)+\pi_1\beta=\pi_0\\ \pi_0\alpha+\pi_1(1{-}\beta)=\pi_1\\ \pi_0{+}\pi_1=1\end{array}\right. \Leftrightarrow \left\{\begin{array}{c}\pi_0(1{-}\alpha)+\pi_1\beta=\pi_0\\ \pi_0{+}\pi_1=1\end{array}\right. \Leftrightarrow \left\{\begin{array}{c}\pi_0(\alpha{+}\beta)=\beta\\ \pi_0{+}\pi_1=1\end{array}\right.,$$

which in turn has a unique solution $\pi=(\frac{\beta}{\alpha+\beta}\;\;\frac{\alpha}{\alpha+\beta})$ if and only if $\alpha+\beta>0$ (whilst for $\alpha+\beta=0$ any distribution π is a solution).