MSG800/MVE170 Basic Stochastic Processes

Written exam Monday 11 January 2016 $2-6$ pm

(With two figures.)

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AIDS: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

GRADES: 12 points for grades 3 and G, 18 points for grade 4, 21 points for grade VG and 24 points for grade 5, respectively.

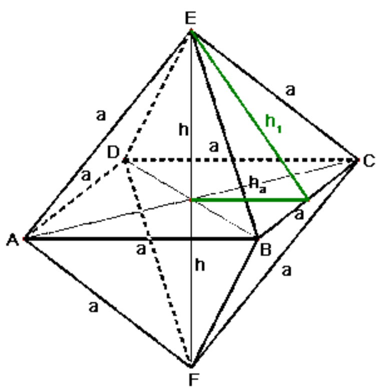
MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Calculate the probability $P(X(1)=1,\ X(4)=4\,|\,X(2)=2,\ X(3)=3,\ X(5)=5)$ for a Poisson process $X(t)$ with arrival rate $\lambda > 0$. **(5 points)**

Task 2. A discrete time martingale M_n, $n \in \mathbb{N}$, is defined by the requirement that $E[M_{n+1}|F_n] = M_n$ for $n \in \mathbb{N}$, where $F_n = \sigma(M_0, \ldots, M_n)$ is the information obtained by observing the values of $M_0, \ldots, M_n$. Prove that this definition implies the seemingly more demanding requirement $E[M_{m+n}|F_n] = M_n$ for $m = 1, 2, 3, \ldots$. **(5 points)**

Task 3. Is it possible for a two state discrete time Markov chain not to have a stationary distribution? **(5 points)**

Task 4. A continuous time random walk on the corners $\{A, B, C, D, E, F\}$ of an octaeder spends a unit mean exponentially distributed random time at each corner after which it selects one of the four neighbour corners as its next position with equal probabilities 1/4. Find the characteristic function for the random time it takes the random walk to move from A to C (see figure below).



(5 points)

Task 5. Let $\{W(t)\}_{t\geq 0}$ be a random process with autocorrelation function $R_{WW}(s,t) = \min\{s,t\}$. Form a new process $Y(t)$ as $Y(t) = \int_0^t W(u)\,du$ for $t \geq 0$. Find the autocorrelation function $R_{YY}(s,t)$ for $0 \leq s \leq t$. **(5 points)**

Task 6. The passport issuing service in former East Germany (=the German Democratic Republic) opened at a certain unpredictable time in the morning each day after which it was open exactly six hours after which it closed down for the day. It was forbidden for passport applicants to queue outside the passport issuing service before the opening time. When the passport issuing service opened each morning passport applicants started to arrive according to a Poisson process with arrival rate 6 applicants per hour. The passport issuing service had just one passport issuer who needed an exponential distributed time with mean 1/2 hour to issue a passport. A passport applicant that was in progress with her/his passport issuing at the closing time was abandoned (didn't get a passport). Write a computer programme that by means of stochastic simulation find an approximation of the expected value of the number of passport applicants that entered the passport issuing service but did not get a passport each day. (Or in other words, find the mean number value of the number of customers that is not being finished served during the first six time units for an M/M/1 queueing system with $\lambda = 6$ and $\mu = 2$ when it is started empty.)

(5 points)

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Solutions to written exam 11 January 2016

Task 1. $P(X(1)=1,\ X(4)=4\,|\,X(2)=2,\ X(3)=3,\ X(5)=5)$

$$= \frac{P(X(1)=1,\ X(4)=4,\ X(2)=2,\ X(3)=3,\ X(5)=5)}{P(X(2)=2,\ X(3)=3,\ X(5)=5)}$$

$$= \frac{P(X(1)=1,\ X(2)-X(1)=1,\ X(3)-X(2)=1,\ X(4)-X(3)=1,\ X(5)-X(4)=1)}{P(X(2)=2,\ X(3)-X(2)=1,\ X(5)-X(3)=2)}$$

$$= \frac{[P(X(1)=1)]^5}{P(X(2)=2)\,P(X(2)=1)\,P(X(2)=2)} = \frac{[\lambda^1/((1!)\cdot \mathrm{e}^{\lambda})]^4}{[(2\lambda)^2/((2!)\cdot \mathrm{e}^{2\lambda})]^2} = \frac{1}{4}.$$

Task 2. This is solved Exercise 5.66 in Hsu's book: See his solution.

Task 3. A stationary distribution π always exists as we can always solve

$$\begin{cases} [\pi_1\ \ \pi_2] = [\pi_1\ \ \pi_2]\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \\ \pi_1+\pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} a\pi_1 = b\pi_2 \\ \pi_1+\pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} \begin{cases} \pi_1 = b/(a+b) \\ \pi_2 = a/(a+b) \end{cases} & \text{if } a+b\neq 0 \\ \pi_1+\pi_2 = 1 & \text{if } a+b=0 \end{cases}.$$

Task 4. We have $\Psi(\omega) = E[\mathrm{e}^{j\omega\exp(1)}]\,E[\mathrm{e}^{j\omega\exp(1/2)}]\,(1/2+(1/2)\cdot\Psi(\omega))$, so that $\Psi(\omega) = E[\mathrm{e}^{j\omega\exp(1)}]\,E[\mathrm{e}^{j\omega\exp(1/2)}]/\big(2-E[\mathrm{e}^{j\omega\exp(1)}]\,E[\mathrm{e}^{j\omega\exp(1/2)}]\big) = \ldots$.

Task 5. $R_{YY}(s,t) = E\big[(\int_0^s W(u)\,du)\,(\int_0^t W(v)\,dv)\big] = \int_{u=0}^{u=s}\int_{v=0}^{v=t} E[W(u)W(v)]\,dudv = \int_{u=0}^{u=s}\int_{v=0}^{v=s} \min\{u,v\}\,dudv + \int_{u=0}^{u=s}\int_{v=s}^{v=t} \min\{u,v\}\,dudv = 2\int_{u=0}^{u=s}\int_{v=0}^{v=u} v\,dvdu + \int_{u=0}^{u=s}\int_{v=s}^{v=t} u\,dvdu = 2\int_0^s u^2/2\,du + \int_0^s (t-s)\,u\,du = \ldots = ts^2/2 - s^3/6.$

Task 6. The sought after average is the expected number of arrivals $6\cdot 6 = 36$ minus the average number of customers that is being served and can be approximated as

```
wiederholungen = 1000000000000000000000000000000; reisepass = 0;
        for durchlauf = 1:wiederholungen
          ankunft = exprnd(1/6); betrieb = ankunft + exprnd(1/2);
          while betrieb < 6
            reisepass = reisepass + 1; ankunft = ankunft + exprnd(1/6);
            if betrieb < ankunft
              betrieb = ankunft + exprnd(1/2);
            else
              betrieb = betrieb + exprnd(1/2);
            end
          end
        end
        36-reisepass/wiederholungen
        >>24.4969526042769871547617952l5
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